PART - III

TEACHING OF MATHEMATICS

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UNIT
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INTRODUCTION

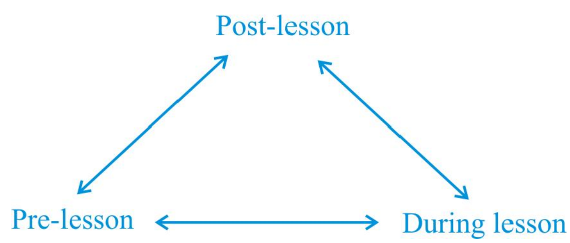
> "A teacher of mathematics has a great opportunity. If he/she fills his/her allotted time with drilling his students is routine operations, he/she kills their interest, hampers their intellectual development and misuses his/her opportunity. But if he/she challanges the curiosity of his/her students by setting them problems proportionals to their knowledge and helps them to solve their problems with stimulating questions, he may give them a taste for, and some means of independent thinking"
>
> ***(G .Polya)***

One argument we often place is that misconceptions in mathematics are a result of ineffective teaching. May be the students have not had enough opportunity to reflect on discussion and mathematical reasoning to support the concept. Effective difficulty yet, is that persists some mathematical mistakes are made all over the world irrespective of Curriculum and Pedagogical strategies adopted. We must understand that learners create their own meanings and structures for mathematics on the basis of their previous experiences.

Teaching mathematics is both a challenging and stimulating endeavour. New insight, new experiences help teachers to examine their belief system about pedagogy they opt for. The abstract nature of mathematics has inevitably lead teachers to re-structure pedagogy and reasouces to help pupils understand mathematical ideas and the underlying conceptual structure. The present module is an attempt to re-visit important mathematical concepts taught at secondary level. After a brief review obtained from the secondary mathematics teachers about some typical misconceptions, and the frequent errors made by students and the probable causes, it was felt important to develop a bridge module to help teachers re-visit these concepts and use more engaging and effective pedagogy.

Mathematics teaching often criticised for its emphasis on memorizing basic facts, rules and formulae. It is always suggested that emphasis should be laid on mathematical reasoning and higher order thinking skills such as application, analysis, synthesis, evaluation and creation (Bloom's revised taxonomy).

Mathematical concepts are abstract in nature and helping learners construct these meaningfully has always been a challenge for teachers. Teaching mathematics requires thinking about concepts, learner centered pedagogy and diversified creative assessment. Teaching act is a three-tire process:



This is not a linear process. Each phase provides linkage and feedback to other two.

The teacher must be congnizant about nature of the learner and nature of the concept. Mathematical ideas are not learnt as the outcome of one lesson. Rather it is an accumilation of previous mathematics experiences. Mathematics anxiety is another factor that contributes to wrongful mathematics learning.

Since mathematics teaching is directly responsible for already learnt mathematics knowledge and making decisions about future mathematics learning, the correct and strong foundations of previously learnt mathematical concepts can help a teacher to plan appropriate strategy to meaningfully expand students' knowledge.

Learner's beliefs towards mathematics must be shaped to generate a positive attitude. Teachers must have a sound knowledge of mathematical concepts to engage students on their understanding of those concepts. Teachers can model different aspects of problem solving and engage students in activities and discussions around the concept.

Effective mathematics teaching should aim to promote students' confidence in mathematics, curiosity, freedom and belief in doing mathematics.

The NCTM (1991) professional teaching standard defines the role of mathematics teachers as:

(a) teach concepts, procedures and convictions.

(b) Promote mathematical problem solving, reasoning and convictions.

(c) foster students' mathematical dispositions.

(d) assess students' understanding of mathematics.

(e) create a learning environment that promotes the development of each child's mathematical power.

Teaching mathematics requires better prepardness to create and help learners create new conceptual structures and the ability to extend existing structures. To do this effectivly teacher must be:

- cognizant of how learner constructs mathematics.
- familiar with the inter-related mathematical concepts.
- well-equipped with multi-pronged strategies and activities that can help children visualize, explore and communicate mathematics knowledge.
- able to assess the level of development and to diagnose remedial plans to rectify misconceptions.

The past few decades have witnessed a resurgence of interest in looking at mathematical errors, that students make and misconceptions they develop during the teaching-learning process. Although re-adderssing misconception has always been an integral part of mathematics programmes, it is always looked in a narrow sense of possible remedial plan. More recently the research has reiterated the importance of misconceptions in understanding learner's cognition, mathematical thinking and inquiry.

The present training module is about teaching-learning considerations for engaging students in meaningful construction of mathematical concepts. It is also about empowring mathematics teachers to devise solution plans from students' responses to re-construct crucial mathematical concepts.

Important Mathematics as Secondary Stage

Being mathematically literate includes having an appreciation of the value and beauty of mathematics as well as being able and inclined to appraise and use quantitative information. Mathematical power encompasses the ability to "explore, conjecture and reason logically, as well as the ability to use a variety of mathematical methods effectively to solve non-routine problems and the self confidence and desposition to do so."

(NCTM, 1989)

To enable students' to develop mathematical power requires a balance between the curriculum and pedagogy to orchestrate classroom discourse to engage students in intellectual activities to promote mathematical ideas.

At secondary level Mathematics comprises of different topical arrangments such as algebra, geometry and probability, but all these strands end up completely interconnected. These interconnections must be effectively woven through resourceful teaching. A coherent curriculum helps to construct and integrate important mathematical ideas to build more refined conceptual structures.

Secondary years are a phase of transition when learners become more ambitious, independent, probing and reflective. Secondary mathematics currimulum should enable students to see the linkage of Algebra, Geometry, Probability, Statistics and to look upon various ways to represent mathematical ideas. They should enhance their abilities to visualise, represent and analyse experiences in mathematical terms through more sophisticated and insightful understnading.

The objectives of secondary mathematics curriculum is to provide students opportunities to be equiped with important mathematics needed for better educational/professional/social choices. It empowers students to investigate, to make sense of and to construct mathematical meanings from new situations.

Secondary mathematics curriculum should provide a roadmap for students to explore their career interests and educational choices.

The purpose of the present module is to strengthen teachers' ability:

- to use appropriate strategies and resources for teaching important topical strands.
- to choose/create worth while mathematical tasks to promote clarity and interest among students.
- to re-construct framework of important mathematical concepts for its inherent, coherance and consistency.
- to use improvised means for assessing students' understanding of mathematics.
- to promote reflection and professional exchange of ideas and experiences among mathematics teachers.

Ideally, students should make sense of what is being taught to them. As students progress from early school years to senior grades, they should develop deeper understanding of numbers as a system of thought, as quantifiers, as a means of communication and representation. A gradual progression in computational fluency to manifest mental strategies and alternate algorithms is important to make students more reasonable and thoughtful.

Algebra is the language of mathematics to communicate mathematical ideas. It is a way to abstract concepts and make generalisations beyond the original context. It enables learners to appreciate the powers of mathematical abstraction, symbolism and generalisation.

Geometry occupies important place in secondary mathematics curriculum where students learn to appreciate axiomatic structure and power of geometrical proofs. A well equipped teacher can help students to explore conjectures and to strengthen logic.

The most significant transition in mathematics is understanding geometry as an algebraic system. The interplay between geometry and algebra strengthens students' ability to visualise, formulate and translate among these systems.

Tringometry as a study of triangle measurement is an indispensible tool to many real world problems from the fields of navigation and surveying. It is based on the precisely

defined ratios of sides and angles of a right angled triangle. These scientifically defined ratios create numerous identities involving plenty of trigonometric applications.

In everyday life, data rules the world to summarise, analyse and transform information. Data collection, organisation, representation and interpretation are important to make meanigful inferences. Statistics as a part of the curriculum should help students to appreciate the differences between mathematical exactness and statistical approximation.

Mathematics as a unified body of inter-related concepts should be a highly-valued subject for students. It requires right attitude of professionally inclined mathematics teachers constantly engaged in reflection practices.

Development of Training Package and Orientation of Master Trainers

National Curriculum Framework-2005 suggests for an overall change in approach to teaching-learning with main focus on constructivist teachniques based on cunstructivist learning theory. This theory believes that learning always builds upon the knowledge the child already has. Learning is more effective when the learner is actively engaged in the learning process rather than when he/she receives the knowledge passively. Textbooks developed by NCERT based on NCF-2005 were written with this shift in emphasis. On teaching of mathematics, NCF-2005 recommends:

- Shifting of the focus from achieving narrow goals relating to numeracy to higher goals of developing a child's inner resources of thinking, clarity of thought and pursuing assumptions to logical conclusions- an ability mathematics teaching needs to create in the learner to handle abstractions. Understanding of when and how a mathematical technique has to be used rather than recalling the technques from memory. Learning of mathematics should be made a part of learner's life experience. Learner should be able to pose and solve meaningful problems. Development of problem solving as a skill during mathematical learning is of great value.
- Engaging every student with a sense of success, while at the same time offering conceptual challenges to the mathematical gifted children.
- Enriching teachers with a variety of mathematical resources.

Consequent upon the development of textbooks based on NCF-2005, identified teachers of Kendriya Vidyalaya Sanghatan, Jawahar Navodaya Vidyalaya Samiti and state of Haryana were trained in the use of these textbooks as master trainers through face to face mode. Some teachers were trained through Teleconferencing mode also. It was expected that the KVS, JNV Samiti and Haryana will arrange further training of teachers using these master trainers. However, this does not seem to have materialised on a large scale.

The present mathematics textbooks at the secondary stage are in use for about 5 years. Now NCERT has also developed books on Exemplar problems as well as Laboratory manuals in Mathemetics for secondary stage. In order to guide the teachers in transacting mathematics at the secondary level, based on implications of NCF-2005, there is a need to

organise traning programmes and hence a need for the development of a traning package to facilitate to conduct these training programmes. This exercise is an attempt to develop a training package and to orient the identified master trainers in using this training package in the training programmes to be organised subsequently by KVS, JNV and various states for its classroom teachers. The training package is envisaged to include strategies for teaching of mathematics in general and a few selected mathematical topics in particular as exemplars for teachers. The training package will be based on the identified training needs of the teachers in teaching-learning process of concepts, generalisations and applications and will also include development of problem solving abilities in the learners as well as Assessment of Mathematics learning.

For the identification of training needs of the teachers, a questionnaire was developed and was sent to the stakeholders. Their responses were analysed and hardspots for transaction of mathematics at the secondary stage were identified. The identified hardspots were discussed threadbare and the training needs were finalised in a 3 day planning meeting during July 29-31, 2011. The design structure and format of the training package were also discussed and finalized. The authors for different units of the package were identified so as to make preliminary preparation for the development of units. The draft of the training package was developed in a workshop mode during October 10-13, 2011. This draft of training package was edited in a 5-day workshop during December 20-24, 2011. Finally, the training was organised w.e.f. Feb. 13-17-2012 in the department, NIE campus, NCERT, New Delhi. The draft was reviewed on the basis of training and edited for the use of RMSA in the financial year 2012-13. The training package is ready now in the present form.

The Training package consists of the following units:

0. Introduction
1. Teaching of the Number Systems
2. Teaching of Algebra
3. Teaching of Geometry
4. Teaching of Mensuration
5. Teaching of Statistics and Probability
6. Problem solving in Mathematics
7. Concept of Educatoinal Evaluation
8. Assessing Mathematics Learning

It is suggested that, while conducting the training programme for the teachers using this training package, the programme should start with one session on NCF 2005 and its recommendations for Mathematics Education and one session on implications of NCF-2005 on teaching of mathematics at secondary stage. These two sessions may be followed by one session each on general strategies for each of the identitied five areas of mathematics and exemplars on teaching of a few of the selected topics in these areas. One sessions each may be kept for development of exempars by the participating teachers on teaching of a few

selected topics in these areas and one sessions each for their presentation and discussion. Two sessions each may be devoted for problem solving in mathematics and assessing mathematics learninig. One session may be for getting the feedback of the participating teachers on the training programme. In view of the above, a tentative schedule is given below for the training programme.

Day	9:00 am to 10:15 am	10:30 am to 11:45 am	12:00 pm to 1:15 pm	2.15 pm to 3:30 pm	3:45 pm to 5:15 pm
Day 1	Registration/ Inauguration	NCF 2005 and its recommendation on Mathematics Education	Imlications of NCF-2005 on Teaching of Mathematics at the Secondary Stage	Teaching of Number systems and exemplars on teaching of a few topics in number systems	Development of exemplars on teaching of selected topics in number systems by participants
Day 2	Problem solving in mathematics	Presentation of exemplars developed by the participants on teaching of number systems	Teaching of algebra and exemplars on teaching of a few topics in algebra	Development of exemplars on teachinig of selected topics in algebra by participants	Presentation of exemplars developed by participants on teaching of algebra
Day 3	Assessing Mathematics Learning-I	Group work on problem solving	Teaching of Geometry and exemplars on teaching of a few topics in Geometry	Development of exemplars on teaching of selected topics in Geometry by participants	Presentation of exemplars developed by participants on teaching of Geometry
Day 4	Teaching of Mathematics and exem-plars on teaching of few topics in Mensuration	Development of exemplars on teaching of selected topics in Mensuration	Presentation of exemplars developed by the participants on teaching of Mensuration	Group work on assessing mathematics learning	
Day 5	Teaching of Statistics and Probability and exem-plars on a few topics in Statistics and Probability	Devlopment of exemplars on teaching of selected topics in Statistics and Probability	Presentation of exemplars developed by the participants on teaching of Statistics and Probability	Evaluation and feedback	

UNIT 1

TEACHING OF NUMBER SYSTEMS

1.1 Introduction

Have you ever thought about the kinds of numbers, we use almost everyday for counting objects? For this purpose, we use the numbers 1, 2, 3 and so on. These numbers are called counting numbers or natural numbers. Did you ever wonder where they came from?

Prehistoric men had little need for numbers. However, as their possessions increased in number and quantity, they thought of crude ways of keeping track of things – ways of recording their belongings. For instance, when a man's sheep went out for grazing in the morning, he would put one stone for each sheep and make a pile. Then at night, when the sheep returned, he would take out one stone for each sheep from that pile. In this way, he would know whether all the sheep returned or not.

After a very long time, instead of matching one object with another, men expressed numbers by various positions of the fingers and hands. But it was not suitable for performing calculations. Later on, with the refinement of writing, an assortment of symbols was evolved to stand for these numbers. For example, Greeks used the letters α, β... the Romans used I, IV, X to denote numbers 1, 4, 10 respectively that we use even now. Finally, the symbols that we use today, the Hindu-Arabic number symbols 1, 2, 3, 4, ..., were invented. The Hindu-Arabic numeral system is named after the Hindus, who invented it, and after the Arabs, who transmitted it to western Europe. The earliest preserved examples of our present number symbols are found in some stone columns erected in India 250 BC by king Ashoka.

In the early times, people used numbers only for counting the few possessions they had, but in the course of time, there came a need for a symbol to express 'not any'. Hindu

scholars invented the symbol '0' (zero) to stand for 'not any' or a symbol to represent an empty collection. The natural numbers along with the number '0' are called *whole numbers*.

What is to be done if a number expresses a situation that means the opposite of the situation represented by a natural number? For example, if a natural number represents 5 degrees above zero, how 5 degrees below zero be indicated? Our forefathers were continually facing problems similar to this. Another important extension of our number system came hundreds of years later. With 0 as a point of origin, it was possible to think of numbers to the left of the point of origin, corresponding to a number to the right of origin. For example –5 was taken corresponding to 5, –3 corresponding to 3 and so on. Such negative numbers, ..., –4, –3, –2, –1, ..., alongwith whole numbers are called *integers*.

Especially while measuring things, man realised his lack of ability to show a part of something, for instance, how much water was in a pitcher if the pitcher was not full. This necessity led man to invent new numbers called *fractions*. A fraction is of the type $\frac{p}{q}$, where p and q are natural numbers.

Like integers, to represent opposite situtations represented by fractions, new numbers such as $\frac{-2}{3}, \frac{-4}{9}, \frac{-5}{7}$, etc., were included in the number system. These new numbers alongwith integers and fractions are called *rational numbers*. Thus, a rational number is of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

As early as the sixth century BC, certain numbers were encountered that were not rational numbers. That is, these numbers could not be expressed as the quotient of two integers. Pythagoras discovered that the length of the diagonal of a square with side one foot long was $\sqrt{2}$ feet long. It is impossible to express it exactly as the quotient of two integers. Such numbers are called *irrational numbers*.

All rational and irrational numbers together constitute a collection what we call the collection of *real numbers*.

Real numbers are used throughout mathematics and we should be acquainted with these numbers, such as

$$2, 54, -3, \frac{26}{7}, \sqrt{3}, 0, \sqrt[3]{74}, 0.444..., 291.38, 1.212112111... \text{ and so on.}$$

1.2 General Strategies

1. Recalling concepts of number system studied earlier.
2. Using inductive-deductive approach to teaching.
3. Explaining various concepts through examples and activities.
4. Enabling students to solve application level problems.

1.3 Key Concepts

- Rational numbers
- Irrational numbers
- Locating irrational numbers on the number line
- Real numbers and their decimal expansions
- Operations on real numbers
- Laws of exponents for real numbers
- Surds
- Rationalisation of denominator
- Proving irrationality of numbers
- Exploring the conditions when the decimal expansion of a rational number is terminating and when non-terminating repeating.
- Euclid's Division Lemma
- Euclid's Division Algorithm to obtain HCF of two positive integers
- Fundamental Theorem of Arithmetic and its applications

1.4 Teaching Strategies

Now we shall discuss a few key concepts listed below for teaching–learning in a classroom situation:

(i) Surds

(ii) Rationalisation of denominator

(iii) Euclid's Division Lemma

Surds

T : You are already familiar with rational and irrational numbers. Can you give examples of some rational and some irrational numbers?

S_1 : 3, –5, 0, $\frac{2}{3}$ and $-\frac{2}{3}$ are rational numbers.

S_2 : $\sqrt{2}, \sqrt{3}, \sqrt[3]{6}, -\sqrt{7}, \sqrt{2}+\sqrt{3}, 2+\sqrt{5}$ and $\sqrt{\frac{2}{3}}$ are irrational numbers.

T : In all these irrational numbers, symbol '$\sqrt{\ }$' has been used. Do you know the name of this symbol?

S_2 : Yes Madam, it is called a 'radical'.

T : Good! You may see that out of these numbers, in the numbers $\sqrt{2}, \sqrt{3}, \sqrt[3]{6}, -\sqrt{7}$ and $\sqrt{\frac{2}{3}}$, there is a rational number under the radical sign '$\sqrt{\ }$'.

S_3 : Madam! Is there any special name for such numbers?

T : Such numbers are called 'Surds', Thus, $\sqrt[n]{a}$ is called a surd, if

(i) $\sqrt[n]{a}$ is an irrational number and

(ii) a is a rational number.

Can you give some more examples of surds?

S : $\sqrt{5}$, $\sqrt{6}$

T : Give some other examples also.

S : $\sqrt[5]{4}$, $\sqrt[3]{5}$, $\sqrt[6]{2}$

T : Very good.

Now give some examples of numbers having radical sign which are not surds.

No response from students!

T : Look at the definition of a surd. If any of the above two conditions is not fulfilled, then the number will not be a surd. First condition is that $\sqrt[n]{a}$ is an irrational number. So, now think of a number of the type $\sqrt[n]{a}$ which is not an irrational number, i.e., a rational number.

S_1 : $\sqrt{4}$

S_2 : $\sqrt{9}$

T : Good. Is $\sqrt[3]{8}$ a surd?

S_1 : Yes.

S_2 : No, Madam, since $\sqrt[3]{8}$ is 2, so it is **not** a surd.

T : Similarly $\sqrt[4]{16}$, $\sqrt[5]{32}$ are also examples of numbers which are not surds.

T : Now look at the second condition that says that *a* in $\sqrt[n]{a}$ is a rational number, So numbers such as

$\sqrt{\pi}$, $\sqrt{2-\sqrt{2}}$ etc., are also not surds.

T : Now what about $\sqrt{\sqrt{4}}$, $\sqrt{\sqrt{16}}$? Are these surds?

S : No.

T : Why?

No response!

T : $\sqrt{\sqrt{4}}$ can be written as $\sqrt{2}$.

and both the conditions of surds are fulfilled here, i.e., $\sqrt{2}$ is an irrational number and 2 is a rational number. So, $\sqrt{\sqrt{4}}$ is a surd.

$\sqrt{\sqrt{16}} = \sqrt{4} = 2$ which is a rational number and so $\sqrt{\sqrt{16}}$ is not a surd.

T : Is $\sqrt{125}$ a surd ?

S : Yes, $\sqrt{125} = \sqrt{5 \times 5 \times 5} = \sqrt{5 \times 5} \times \sqrt{5} = 5\sqrt{5}$, which is a surd as $5\sqrt{5}$ is irrational.

T : So, $\sqrt{125} = 5\sqrt{5}$. Writing $\sqrt{125}$ in the term $5\sqrt{5}$ is called simplifying a surd.

Now simplify $\sqrt{18}$.

S : $\sqrt{18} = \sqrt{3 \times 3 \times 2} = \sqrt{3 \times 3} \times \sqrt{2} = 3\sqrt{2}$

T : Good.

What is $7\sqrt{2} - 4\sqrt{2}$?

S : It is $3\sqrt{2}$.

T : Can you simplify $16\sqrt{5} - 3\sqrt{125}$?

S : We cannot simplify it.

T : Why?

S : Because both terms are different.

T : Can you simplify $\sqrt{125}$?

S : Yes. We have done it just now. It is $5\sqrt{5}$.

Oh yes! We can simplify $16\sqrt{5}-3\sqrt{125}$ as

$16\sqrt{5}-3\times 5\sqrt{5}=16\sqrt{5}-15\sqrt{5}=\sqrt{5}$

T : Very good.

What is $\sqrt{10}\times\sqrt{5}$?

S : It is $\sqrt{2\times 5}\times\sqrt{5}$

and stops here.

T : Yes! Yes!! Go on

S : $=\sqrt{2\times 5\times 5}=\sqrt{2}\times\sqrt{5\times 5}=\sqrt{2}\times 5=5\sqrt{2}$

T : Very good.

T : What is $2\sqrt{2}\times 3\sqrt{2}$?

S : It is $2\times 3\sqrt{2}=6\sqrt{2}$

T : Are you sure? What is $2x\times 3x$?

S : $6x^2$

T : So, how $2\sqrt{2}\times 3\sqrt{2}$ is $2\times 3\sqrt{2}$?

S : Oh yes! It should be $2\times 3\times\left(\sqrt{2}\right)^2=6\times 2=12$

T : Right.

Now tell me what is $\sqrt{7}\left(\sqrt{14}+\sqrt{7}\right)$?

S : It will be $\sqrt{7}\ \sqrt{14}+7$

The student stops here a little and then says Madam! we can simplify it further.

T : Yes, do it.

S : $\sqrt{7}\sqrt{2\times 7}+7=\sqrt{7}\times\sqrt{2}\times\sqrt{7}+7=7\sqrt{2}+7$

T : Yes. O.K.

What is the value of $\left(1+\sqrt{3}\right)\left(1-\sqrt{3}\right)$?

S : It will be $1+\sqrt{3}-\sqrt{3}-\sqrt{3}\sqrt{3}=1-3=-2$

T : Can you use some formula (algebraic identity) to simplify it?

S : Oh yes! We can.

T : What is that formula?

S : $(a+b)(a-b)=a^2-b^2$

So, it will be $1^2-\left(\sqrt{3}\right)^2=1-3=-2$

T : Good.

Now, simplify $\left(2\sqrt{10}+3\sqrt{5}\right)\left(2\sqrt{10}-3\sqrt{5}\right)$

S : It will be $2\sqrt{10^2}-3\sqrt{5^2}$

$=2\times 10-3\times 5=5.$

T : Is it correct?

S_2 : Madam! It should be $\left(2\sqrt{10}\right)^2-\left(3\sqrt{5}\right)^2$

$=4\times 10-9\times 5=40-45=-5$

T : Yes, this is the right way.

Simplify $\left(1+\sqrt{2}\right)\left(\sqrt{3}-1\right)-2\sqrt{5}$

S : $\sqrt{3}-1+\sqrt{2}\sqrt{3}-\sqrt{2}-2\sqrt{5}$

S : It can be further simplified as $\sqrt{3}-1+\sqrt{6}-\sqrt{2}-2\sqrt{5}$

T : Can't we simplify it further?

S : No.

T : Why?

S : Because all the terms are different.

T : Yes, all the terms are unlike terms. Terms which contain unlike surds are called **unlike terms**.

$2\sqrt{3}$, $4\sqrt{5}$, $-\sqrt{2}$ are unlike terms whereas $\sqrt{3}$, $-2\sqrt{3}$, $\frac{3}{4}\sqrt{3}$ are like terms.

What is $\frac{\sqrt{5}}{\sqrt{7}}$?

S : It is $\frac{5}{7}$

T : How did you get it?

S : By cancelling the square root sign from both the numerator and denominator.

T : You cannot cancel square root sign like this. In fact,

$$\frac{\sqrt{5}}{\sqrt{7}} = \frac{5^{1/2}}{7^{1/2}}$$

Recall the rules of exponents. You cannot cancel the exponents of different numbers.

At the most, we can write it as $\sqrt{\frac{5}{7}}$.

Similarly, $\frac{\sqrt{8}}{\sqrt{2}} \neq \frac{8}{2} = 4$

It is $\sqrt{\frac{8}{2}} = \sqrt{4} = 2$

Review Questions

Simplify the following :

(i) $\sqrt{150}$

(ii) $\sqrt{112}$

(iii) $\sqrt{10} \times \sqrt{5}$

(iv) $\left(5\sqrt{5}\right)^2$

(v) $9\sqrt{5} - 2\sqrt{5}$

(vi) $26\sqrt{2} + 10\sqrt{8}$

(vii) $\frac{7\sqrt{2}}{2\sqrt{2}}$

(viii) $\left(2+\sqrt{7}\right)\left(\sqrt{7}-2\right)$

(ix) $\left(2\sqrt{10}+3\sqrt{5}\right)\left(2\sqrt{10}-3\sqrt{5}\right)$

(ii) Rationalisation of Denominator

T : Can you find the value of $\frac{1}{\sqrt{3}}$, given $\sqrt{3} = 1.732$

S_1 : $\frac{1}{\sqrt{3}} = \frac{1}{1.732} = \frac{1000}{1732}$

$$
\begin{array}{r}
0.577 \\
1732\overline{)\,1000.000} \\
\underline{866\ 0} \\
134\ 00 \\
\underline{121\ 24} \\
12\ \ 760 \\
\underline{12\ \ 124} \\
636
\end{array}
$$

Thus, $\frac{1}{\sqrt{3}} = 0.577$

S$_2$: I can do it like this: If we write $\frac{1}{\sqrt{3}}$ as

$$\frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3} = \frac{1.732}{3} = 0.577$$

T : Both the methods are correct, but in second method, by multiplying numerator and denominator by $\sqrt{3}$, the denominator becomes a rational number. By putting the value of $\sqrt{3}$, you can easily find its value.

Here you can do this division orally whereas in first method, the division was quite tedius.

The technique of removing the surds from the denominator is called *rationalising the denominator*.

T : Now, can you rationalise the denominator of $\frac{10\sqrt{2}}{\sqrt{5}}$?

S : $\frac{10\sqrt{2}}{\sqrt{5}} = \frac{10\sqrt{2}}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{10\sqrt{10}}{5}$

T : Can you further simplify it?

S : Yes madam! I can do it as $\frac{10\sqrt{10}}{5} = 2\sqrt{10}$

T : Good.

Now rationalise the denominator of $\frac{\sqrt{5}}{3\sqrt{7}}$.

S : $\frac{\sqrt{5}}{3\sqrt{7}} = \frac{\sqrt{5}}{3\sqrt{7}} \times \frac{3\sqrt{7}}{3\sqrt{7}} = \frac{3\sqrt{35}}{9 \times 7} = \frac{\sqrt{35}}{21}$

T : Look, in the denominator 3 is already a rational number, so there is no need to multiply and divide with $3\sqrt{7}$. Only $\sqrt{7}$ will be sufficient.

S : Yes, $\frac{\sqrt{5}}{3\sqrt{7}} = \frac{\sqrt{5}}{3\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}} = \frac{\sqrt{5 \times 7}}{3 \times 7} = \frac{\sqrt{35}}{21}$

T : Good. Try to rationalise the denominator of $\frac{1}{3-\sqrt{5}}$.

S_1 : $\frac{1}{3-\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{3\sqrt{5}-5}$

S_2 : Denominator still contains $\sqrt{5}$. It is not rationalised.

S_3 : I have an idea. I can do it as follows:

$$\frac{1}{3-\sqrt{5}} \times \frac{(3+\sqrt{5})}{(3+\sqrt{5})}$$

$$= \frac{3+\sqrt{5}}{3^2-(\sqrt{5})^2}$$

$$= \frac{3+\sqrt{5}}{9-5}$$

$$= \frac{3+\sqrt{5}}{4}$$

T : How did you think of doing it in this way?

S : Madam, I was having the identity $(a - b)(a + b) = (a^2 - b^2)$ in my mind.

T : Now try to rationalise the denominator of $\frac{1}{\sqrt{2}-\sqrt{5}}$.

With what number will you multiply and divide it?

S : If we multiply with $\sqrt{2}$, then surd still remains and if we multiply with $\sqrt{5}$, then again surd will remain.

S looks towards teacher.

T : In this question, if you multiply $\sqrt{2}-\sqrt{5}$ with $\sqrt{2}+\sqrt{5}$, then you get

$\left(\sqrt{2}-\sqrt{5}\right)\left(\sqrt{2}+\sqrt{5}\right)=\left(\sqrt{2}\right)^2-\left(\sqrt{5}\right)^2=2-5=-3$ which is a rational number.

So, $\dfrac{1}{\sqrt{2}-\sqrt{5}}=\dfrac{1}{\sqrt{2}-\sqrt{5}}\times\dfrac{\sqrt{2}+\sqrt{5}}{\sqrt{2}+\sqrt{5}}$

What is the next step?

S : $\dfrac{\sqrt{2}+\sqrt{5}}{(\sqrt{2})^2-(\sqrt{5})^2}=\dfrac{\sqrt{2}+\sqrt{5}}{2-5}=\dfrac{\sqrt{2}+\sqrt{5}}{-3}$

T : Good, but you could write the answer as $\dfrac{-\left(\sqrt{2}+\sqrt{5}\right)}{3}$, so that the denominator becomes positive.

Now we take some more examples.

Rationalise the denominator of $\dfrac{1}{2\sqrt{2}+\sqrt{6}}$.

S : $\dfrac{1}{2\sqrt{2}+\sqrt{6}}=\dfrac{1}{2\sqrt{2}+\sqrt{6}}\times\dfrac{2\sqrt{2}-\sqrt{6}}{2\sqrt{2}-\sqrt{6}}$

$$=\frac{2\sqrt{2}-\sqrt{6}}{\left(2\sqrt{2}\right)^2-\left(\sqrt{6}\right)^2}=\frac{2\sqrt{2}-\sqrt{6}}{8-6}=\frac{2\sqrt{2}-\sqrt{6}}{2}$$

$$=\sqrt{2}-\sqrt{6}$$

Madam! I have written it in the simplified form.

T : It is good that you have thought of simplifying, but you have made a mistake. Check again.

S : $\dfrac{2\sqrt{2}-\sqrt{6}}{2}$ cannot be simplified further. We cannot cancel 2 here.

T : Good. We can cancel 2 only if numerator were of the type $2\left(\sqrt{2}-\sqrt{6}\right)$. But it is not so here.

S : Madam! We have enjoyed the lesson. We have made so many mistakes. We knew all the correct answers, but we don't know how, while applying them we made such mistakes. So give us more questions.

T : Yes, I will give you more questions for practice.

Review Questions

Rationalise the denominator of the following:

(i) $\dfrac{\sqrt{18}}{\sqrt{15}}$

(ii) $\dfrac{1}{3\sqrt{3}}$

(iii) $\dfrac{5+\sqrt{2}}{3\sqrt{2}}$

(iv) $\dfrac{16}{\sqrt{41}-5}$

(v) $\dfrac{3+\sqrt{5}}{2-\sqrt{5}}$

(vi) $\dfrac{5+2\sqrt{3}}{7+4\sqrt{3}}$

(vii) $\dfrac{13}{3\sqrt{6}-2}$

(iii) Euclid's Division Lemma

T : Students!! Divide 17 by 5.

S :
$$\begin{array}{r} 3 \\ 5\overline{)17} \\ \underline{15} \\ 2 \end{array}$$

T : What is the quotient?

S : 3

T : What is the remainder?

S : 2

T : What are 17 and 5 called?

S : 17 is the dividend and 5 the divisor.

T : Do you observe any relation between 5, 17, 3 and 2?

No response from students.

T : We can write 17 as $17 = 5 \times 3 + 2$

i.e., Dividend = Divisor × Quotient + Remainder

T : Divide 21 by 5 and write the process in the above form.

S : $\begin{array}{r} 4 \\ 5\overline{)21} \\ \underline{20} \\ 1 \end{array}$

So, $21 = 5 \times 4 + 1$

T : Now divide 20 by 5.

S : $\begin{array}{r} 4 \\ 5\overline{)20} \\ \underline{20} \\ 0 \end{array}$

So, $20 = 5 \times 4 + 0$

T : Good!

Thus, if a and b are any two positive integers and we divide a by b, then write in the above form when quotient is q and remainder r.

S : $a = bq + r$

T : Very good.

T : If we divide 10, 11, 12, 13, 14, 15, 16, 17 by 5 what remainders do we obtain?

S : The remainders are 0, 1, 2, 3, 4, 0, 1, 2 respectively.

T : Can we have any other remainders for these divisions?

S : No.

T : So, we can say that the remainder in a particular division is unique.

We see that remainders have started repeating after 4. We conclude that when we divide a number by 5 the possible remainders are 0 or 1 or 2 or 3 or 4.

T : Can we have remainder 5 or 6 here?

S : No.

T : So what is the maximum value that the remainder can take here?

S : 4

T : What is the minimum value of the remainder here?

S : 0

T : When we divide 12, 13, 14, 15, 16, 17, 18, 19, 20 by 6, what are the remainders? They are:

S : 0, 1, 2, 3, 4, 5, 0, 1, 2 respectively.

T : Here also, we observe that on division by 6 the possible remainders are 0 or 1 or 2 or 3 or 4 or 5 and cannot have any other remainder except these.

T : What are the maximum and minimum values of the remainder?

S : Maximum value is 5 and the minimum value is 0.

T : So, we conclude from the two examples above that 0 ≤ remainder < the number by which we are dividing

i.e., 0 ≤ remainder < divisor

In general, when we divide a by b and if q is the quotient and r the remainder, then we have already seen that

$a = bq + r$

What is the condition on r ?

S : 0 £ r < divisor (b)

T : So we get

0 £ $r < b$

T : In the above divisions, could we have remainders other than those obtained?

S : No.

T : Thus, we conclude that the remainder in a particular division is unique.

T : All of you divide 128 by 5 and tell me the quotient.

S_1 : It is 25

S_2 : 25

S_3 : 25

T : Raise your hands who have obtained the quotient as 25.

All the students raised their hands.

T : So, answer obtained by all of you is 25. Therefore, we can say that we cannot obtain any other quotient.

T : In the division of 2180 by 9, what is the quotient?

S : It is 242.

T : Has anyone of you obtained the quotient other than 242?

S : No.

T : Thus, we conclude that quotient is also unique.

Thus, if a and b are any two positive integers, then there exists unique integers q and r satisfying

$a = bq + r,\ 0 \le r < b$

This result is known as **Euclid's Division Lemma**.

T : What will be the quotient when we divide 5 by 12 ?

S : We cannot divide 5 by 12.

T : In such cases when the dividend is smaller than the divisor, we can say that the quotient is 0.

Thus, in $a = bq + r, 0 \le r < b;$

we note that q can be zero (0) also.

T : How will you write this division in the form $a = bq + r$?

S_1 : $12 = 5 \times 0 +$ __.

S_2 : Mam! It should be

$5 = 12 \times 0$

as we are dividing 5 by 12 and not 12 by 5

T : Good. You are correct, but give the complete answer.

S_2 : $5 = 12 \times 0 + 5$

T : Very good.

Review Questions

1. Euclid's Division Lemma states that for two positive integers a and b, there exists unique integers q and r such that $a = bq + r$ where r must satisfy :

   (A) $0 < r < b$ (C) $0 < r \le b$

   (B) $0 \le r < b$ (D) $0 < r <$ b

2. The values of the remainder r when a positive integer a is divided by 3 are 0 and 1 only. Whether the statement is true or false. Justify your answer.

1.5 Some Examples with Alternate Solutions

Example I.

T : I am giving you one question of converting $0.2\overline{35}$ in the form $\frac{p}{q}$. Who will come on the black-board and try?

S$_1$: I will try on the board.

Let $x = 0.2\overline{35} = 0.23535K$

We will muliply it with 100 as two digits are repeating.

So, $100x = 23.535K$

We subtract x from this.

$$\begin{array}{lcl} 100x & = & 23.535... \\ -\ x & = & -\ 0.235... \\ \hline 99x & = & \end{array}$$

S$_1$ stops here.

S$_2$: I can further do it now. $99\,x = 23.3$

Thus, $x = \dfrac{23.3}{99}$

i.e., $0.2\overline{35} = \dfrac{23.3}{99}$

T : We have to write the answer in the form $\frac{p}{q}$, where p and q are integers. So, we should remove the decimal point from the numerator.

So, what is the answer?

S : $0.2\overline{35} = \dfrac{233}{990}$

T : I give you another way of solving it.

Let $x = 0.2\overline{35}$

Let us multiply both sides by 10, because only one digit 2 is not repeating. [If such non-repeating digits are two, then we multiply by 100 and so on]

$10x = 2.\overline{35} = 2.3535...... = y$

Now $10x$, i.e., y has been reduced to the ordinary type.

Now who will come on the board to complete it?

S : I will do it.

We will multiply with 100 as two digits are repeating.

So, $100\,y = 235.3535...$

T : What should we subtract from this? x or y?

S : y.

T : Good. Complete it.

S :

$$\begin{array}{rcl} 100y & = & 235.3535... \\ -y & = & -2.3535... \\ \hline 99y & = & 233 \end{array}$$

So, $y = \dfrac{233}{99}$

S stops here

T : Here we have obtained the value of y, but we have to give the value of x. So, we should write

$$y = 10x = \frac{233}{99}$$

i.e., $x = \dfrac{233}{990}$ or $0.2\overline{35} = \dfrac{233}{990}$

S : Yes madam! It is simple. I have understood it.

Example II.

T : After how many decimal places will the decimal expansion of the number $\dfrac{91}{1250}$ terminate?

S :

$$\begin{array}{r|l} & 0.0728 \\ \hline 1250 & 91.0000 \\ & 87\ 50 \\ \hline & 3500 \\ & 2500 \\ \hline & 10000 \\ & 10000 \\ \hline & \times \\ \hline \end{array}$$

So, the decimal expansion of the number has terminated after 4 places of decimal.

T : The answer is correct.

We can give the answer in another way also without actually performing the long division. We already know how to find whether the decimal expansion of $\frac{91}{1250}$ is terminating or not. Let us examine it first. Who will explain it?

S : I will do it.

91 and 1250 are coprime.

Prime factorisation of 1250 is $2 \times 5 \times 5 \times 5 \times 5$

i.e., 2×5^4

So, $\frac{91}{1250} = \frac{91}{2 \times 5^4}$

Since prime factors of the denominator contain only 2 and 5, the decimal expansion is terminating.

T : Good.

Now, if we multiply numerator and denominator by 2^3, the number becomes

$\frac{91 \times 2^3}{2^4 \times 5^4}$, i.e., $\frac{91 \times 8}{(2 \times 5)^4}$, i.e., $\frac{91 \times 8}{10^4}$

Now, can you tell me the decimal places it will have in its expansion?

No response!!

T : Four, because it is simple division by 10^4,i.e., 10000 which can be done orally.

T : Why did we multiply with 2^3 ?

S : To make 2 as 2^4.

T : Good! But why we wanted 2^4?

S : So, that the powers of 2 and 5 become equal.

T : Why we want to make equal powers of 2 and 5?

S : $2^4 \times 5^4$ will make $(2 \times 5)^4$, i.e., 10^4 and division by 10^4 can be done orally.

T : Very good.

Thus, in general

If the number is of the term $\frac{a}{2^m \times 5^n}$ and if

(i) $m < n$, then we will multiply the numerator and denominator with 2^{n-m} to get denominator containing power of 10 only.

So, $\dfrac{a}{2^m \times 5^n} = \dfrac{a \times 2^{n-m}}{2^m \times 5^n \times 2^{n-m}} = \dfrac{a \times 2^{n-m}}{2^n \times 5^n} = \dfrac{a \times 2^{n-m}}{(2 \times 5)^n} = \dfrac{a \times 2^{n-m}}{10^n}$

So, the decimal expansion will terminate after n places of decimal.

(ii) $m > n$, then we will multiply the numerator and denominator with 5^{m-n} to get denominator containing power of 10 only.

So, $\dfrac{a}{2^m \times 5^n} = \dfrac{a \times 5^{m-n}}{2^m \times 5^n \times 5^{m-n}} = \dfrac{a \times 5^{m-n}}{2^m \times 5^m} = \dfrac{a \times 5^{m-n}}{(2 \times 5)^m} = \dfrac{a \times 5^{m-n}}{10^m}$

Thus, the decimal expansion will terminate after m places of decimal.

Thus, we conclude that greater of the two m or n decides the number of places after which the decimal expansion terminates.

1.6 Enrichment Material

If a number has non terminating decimal expansion, after how many places of decimal it must start repeating?

To understand this, let us take some examples. Consider $\dfrac{1}{7}$.

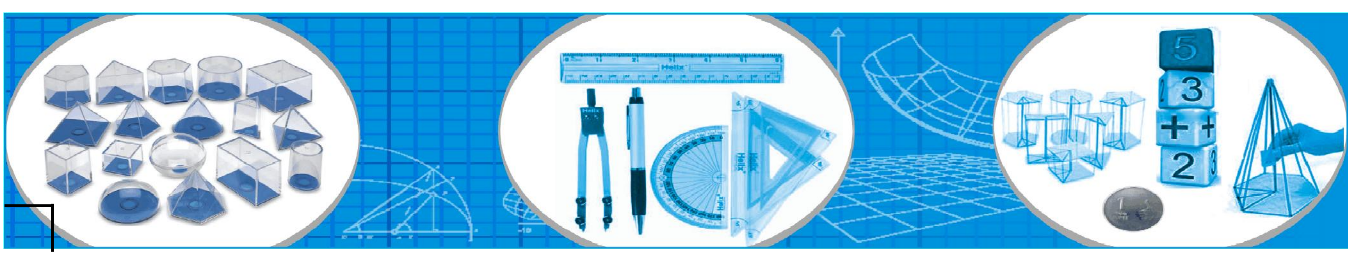
Since 1 and 7 are coprime and the prime factors of the denominator have neither 2 nor 5, the number has non-terminating decimal expansion.

When we divide 1 by 7, by Euclid's Division Lemma, remainder can be 0 or 1 or 2 or 3 or 4 or 5 or 6 but not more than 6. Since the number has non-terminating decimal expansion, remainder cannot be zero and it has to start repeating the values. So, the decimal expansion must repeat after 6 or less than 6 decimal places. i.e., the decimal expansion must start repeating after (denominator –1) places or less than (denominator –1) places when numerator and denominator are coprime. It may repeat after 1 places, 2 places, ..., at the most (denominator –1) places.

Similarly, for the number $\dfrac{2}{17}$, 2 and 17 are coprime and the prime factors of 17 do not have 2 or 5, so the decimal expansion is non-terminating. It must start repeating after at the most 16 places of decimal. It may start repeating earlier.

1.7 Misconceptions/Common Errors

1. $\dfrac{\sqrt{12}}{\sqrt{3}}$ is not a rational number as numerator and denominator are not integers.



Here, $\dfrac{\sqrt{12}}{\sqrt{3}} = \sqrt{\dfrac{12}{3}} = \sqrt{4} = 2$, a rational number.

2. $\sqrt{2}\sqrt{3} = \sqrt{6}$,

   $\sqrt{5}\sqrt{8} = \sqrt{40}$

   So, $\sqrt{a}\ \sqrt{b} = \sqrt{ab}$, for all a, b.

   This relation is true only when a and b are positive numbers.

3. If x is a rational number and y an irrational number, then xy is irrational.

   It is not true, if $x = 0$.

   For other values of x, the result is true.

4. $\sqrt{-4} = -\sqrt{4} = -2$.

   $\sqrt{-4}$ is not defined, whereas $-\sqrt{4}$ is defined. So, $\sqrt{-4} \neq -\sqrt{4}$

5. $\sqrt{a^2 + b^2} = a + b$ or $\sqrt{a + b} = \sqrt{a} + \sqrt{b}$

   These are not true as

   $\sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$, whereas $3 + 4 = 7$

   Thus, $\sqrt{3^2 + 4^2} \neq 3 + 4$.

6. π is a rational number as $\pi = \dfrac{22}{7}$. But it is not so. $\dfrac{22}{7}$ is a crude approximate value of π taken for calculation purpose. In fact, π is an irrational number.

7. π is the ratio of circumference (c) and diameter (d) of a circle, i.e., $\pi = \dfrac{c}{d}$.

   So, π is a rational number as it is in the form $\dfrac{p}{q}$.

   But it is not so as when circumference and diameter of a circle are measured both are not always integers. Thus, π is not a rational number.

8. Decimal expansion of $\sqrt{2} = 1.4142$ which is terminating, so $\sqrt{2}$ is a rational number. It is a misconception as the value 1.4142 is an approximate value of $\sqrt{2}$.

9. Students consider the numbers of the type 1.030030003... as repeating type, i.e., non-terminating repeating and so is a rational number, whereas the fact is that the

number is not a repeating type, because same block of numbers is not repeating here.

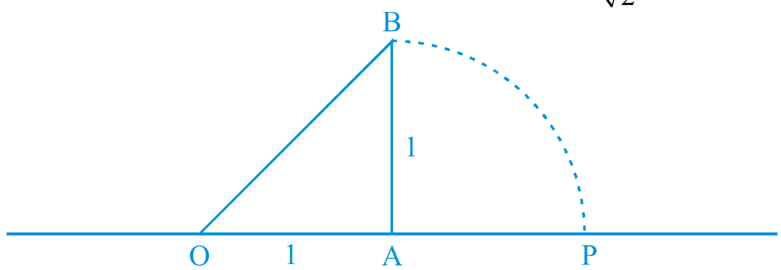
10. Between 1 and 2, the only rational numbers are 1.1, 1.2, 1.3, ..., 1.9. Of course these are some rational numbers between 1 and 2, but there are infinite rational numbers between 1 and 2 like 1.11, 1.1141, 1.1238 etc.

11. There are no irrational numbers between $\sqrt{2}$ and $\sqrt{3}$.

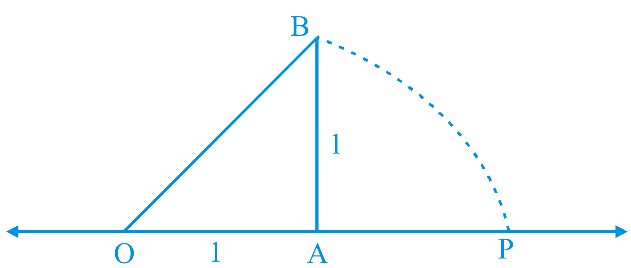
    In fact, there are infinite irrational numbers between $\sqrt{2}$ and $\sqrt{3}$.

12. To locate $\sqrt{2}$ on the number line, some students take

    (1) A as centre and radius = AB and conclude that AP = $\sqrt{2}$.

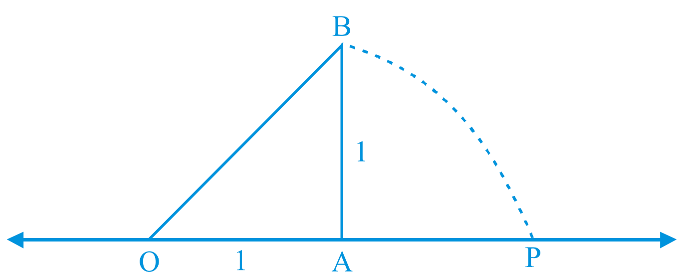


(2) A as centre and radius = OB and conclude $OP = \sqrt{2}$.

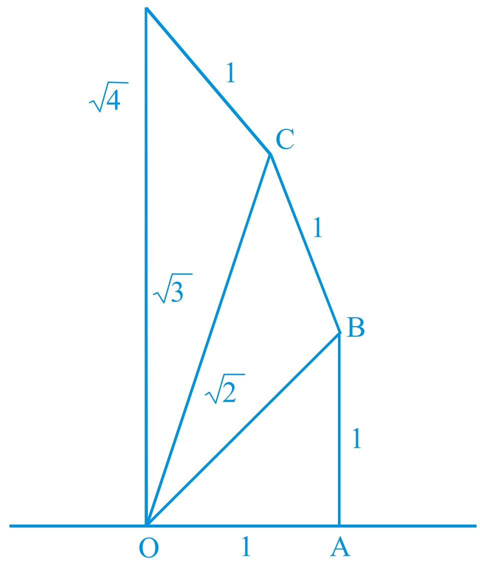


whereas the correct answer is:

Take O as centre and OB as radius and conclude OP = $\sqrt{2}$.



13. In drawing a square root spiral, right angles are not drawn at A, B and C.

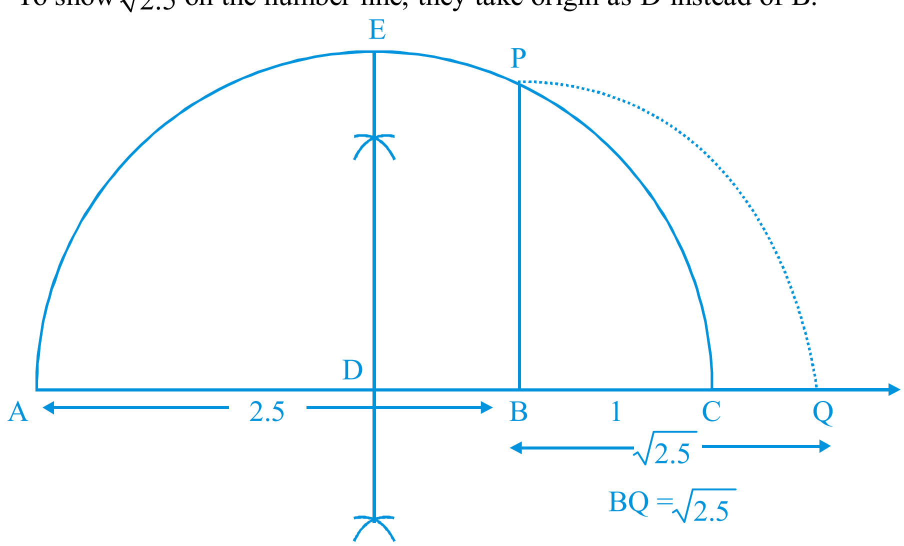


14. To show $\sqrt{2.5}$ on the number line:

(1) Instead of drawing perpendicular bisector of AC, students draw perpendicular bisector of AB.

(2) Instead of saying BP = $\sqrt{2.5}$, they say DE = $\sqrt{2.5}$

(3) To show $\sqrt{2.5}$ on the number line, they take origin as D instead of B.



15. HCF $(p, q) \times$ LCM $(p, q) = p \times q$

For three numbers also, the result is incorrectly used as

HCF $(p, q, r) \times$ LCM $(p, q, r) = p \times q \times r$

The correct results are:

$$\text{LCM}(p, q, r) = \frac{pqr \ \text{HCF}(p,q,r)}{\text{HCF}(p,q)\,.\,\text{HCF}(q,r)\,.\,\text{HCF}(p,r)}$$

$$\text{HCF}(p, q, r) = \frac{pqr \ \text{LCM}(p,q,r)}{\text{LCM}(p,q)\,.\,\text{LCM}(q,r)\,.\,\text{LCM}(p,r)}$$

16. It is misunderstood that the prime factors of denominator should have both 2 and 5 for the decimal expansion to be terminating. If any one of 2 or 5 occurs, then also it has terminating decimal expansion.

17. Euclid's Division Lemma and Euclid's Division Algorithm are taken to be the same.

Lemma is a proven statement whereas algorithm is a stepwise procedure for solving a problem.

18. There are some questions like the following asked by the students :

LCM of two numbers is 84 and their HCF = 5. If one of the numbers is 12, find the other.

Students solve it in the following way :

HCF $\times$ LCM = Product of two numbers.

$5 \times 84 = 12 \times x$

So, $x = \frac{5 \times 84}{12} = 35$

i.e., the other number is 35.

So, the two numbers are 12 and 35.

But in fact, if we find the LCM and HCF of 12 and 35, we do not obtain LCM as 84 and HCF as 5. It is because the question itself was not correct, although the procedure is correct. HCF has to be a factor of LCM, whereas here 5 is not a factor of 84.

19. Some students decide about a number $\frac{p}{q}$ to have terminating or non-terminating expansion just by looking at the denominator without making p and q coprime.

For example, in $\frac{99}{36}$

$36 = 2^2 \times 3^2$

As the prime factors of 36 contain numbers other than 2 and 5, i.e., 3^2, the decimal expansion is non-terminating.

Whereas the correct answer is that

$\frac{99}{36} = \frac{11}{4}$ has terminating decimal expansion as $4 = 2 \times 2$

20. After finding the decimal expansion upto 5 or 6 places in $\frac{1}{17}$, the students may conclude that the decimal expansion is not repeating.

    But it is not so.

    They should understand that $\frac{1}{17}$ is a rational number as it is in the form $\frac{p}{q}$. So its decimal expansion must be either terminating or non-terminating repeating. So, if it is not terminating, it must be repeating.

    Here we are dividing by 17 and so by Euclid's Division Lemma, remainder can be 0 or 1 or 2 or ...... or 16. So, they should continue the division process upto 16 places of decimal to conclude the answer.

1.8 Exercise

Multiple Choice Questions

Write the correct answer :

1. $\left(2^5\right)^{-3}$ is not equal to

   (A) 2^{5-3} (B) $\frac{1}{2^{15}}$ (C) $\frac{1}{\left(2^5\right)^3}$ (D) $\frac{1}{\left(2^3\right)^5}$

2. The number obtained on rationalising the denominator of $\frac{1}{\sqrt{3}-2}$ is

   (A) $\sqrt{3}+2$ (B) $\sqrt{3}-2$

   (C) $-2-\sqrt{3}$ (D) $2-\sqrt{3}$

3. LCM of the smallest composite number and the smallest prime number is

   (A) 0 (B) 1 (C) 2 (D) 4

4. After how many decimal places will the decimal expansion of the number $\frac{47}{2^3 5^2}$ terminate?

   (A) 5 (B) 3 (C) 2 (D) 1

5. The largest number which divides 70 and 125 leaving remainders 5 and 8 respectively is

   (A) 13 (B) 65 (C) 875 (D) 1750

6. $\left(\sqrt{2}\right)^2 = 2$

   $\left(\sqrt{3}\right)^2 = 3$

   So, can we say that square of an irrational number is always a rational number? Justify.

7. Rationalise the denominator of $\frac{\sqrt{2}+\sqrt{3}}{\sqrt{2}-\sqrt{3}}$

8. If $a = 2-\sqrt{5}$, then find the value of $a+\frac{1}{a}$.

9. Is there any natural number n for which 4^n ends with the digit 0? Give reasons in support of your answer.

10. Decimal representation of $\frac{441}{2^2 5^7 7^2}$ is non-terminating. Write true or false justifying your answer.

11. Show that the square of any positive integer can be of the form $4q$ or $4q + 1$ for some integer q.

12. Use Euclid's division algorithm to find the HCF of 441, 567 and 693.

13. Prove that $\sqrt{3}+\sqrt{5}$ is irrational.

14. Show that the square of an odd positive integer can be of the form $6q + 1$ or $6q + 3$ for some integer q.

15. Show that the cube of any positive integer can be of the form $4m$, $4m + 1$ or $4m + 3$ for some integer m.

UNIT 2

TEACHING OF ALGEBRA

2.1 Introduction

Students are first introduced to algebra in the upper primary stage itself. It is indeed here at this stage that transition from arithmetic to algebra takes place. Moving from number sense to number patterns, seeing relationship between numbers and seeking generalisations lead to the introduction of algebraic identities. Necessity for solving daily life problems using compact language gives rise to the introduction of 'variables' or 'unknowns' leading to the development of algebraic expressions, polynomials, linear and quadratic equations and their solutions. By learning **Algebra**, students get exposed to the abstract nature of mathematics. Algebraic manipulations are very important for finding solutions to many problems. The proofs in geometry and trigonometry use algebraic techniques. They demonstrate the power of algebraic techniques to the students.

2.2 General Strategies

1. Motivating from the physical world/life situations.
2. Using inductive-deductive approach to teaching-learning.
3. Explaining with examples.
4. Asking 'why' and 'how' questions/seeking reasons for the responses.
5. Developing algebraic manipulation skills.
6. Enabling students to solve application type problems.
7. Developing alternative approaches to solve problems and contrast the approaches.

2.3 Key Concepts

- Polynomials in one variable.
- Zeroes of a polynomial.
- Relationship between zeroes and coefficients of a polynomial.
- Factorisation of polynomials.
- Algebraic identities.
- Division of one polynomial by another polynomial and division algorithm.
- Linear equations in two variables.
- Solution of linear equations in two variables.
- Pair of linear equations in two variables.
- Graphical method of solution of a pair of linear equations.
- Solution of a pair of linear equations by –
  (i) method of substitution,
  (ii) method of elimination and
  (iii) method of cross multiplication.
- Quadratic equations.
- Solution of quadratic equations by factorisation.
- Solution of quadratic equations by the method of completing the square.
- Equations reducible to a pair of linear equations and quadratic equations.
- Arithmetic progressions.

2.4 Teaching Strategies

We will discuss how a few of the above key concepts have to be transacted in the classroom. They are as follows:

(i) Polynomials, (ii) Zeroes of a polynomial, (iii) Solution of quadratic equations by the method of completing the square and (iv) pair of linear equations in two variables.

(i) Polynomials

Teacher (T) : Students! We have already learnt about variables in our previous classes. Recall that a variable takes different values. For example, temperature on different days takes different values and hence is a variable. Marks scored in

Mathematics by the students in a class are different and hence is a variable. If we consider rectangles, length of each of these rectangles takes different values and hence is a **variable**. Similarly, breadth of rectangles is also a variable.

What can you say about the length of sides of squares ?

Student(S): It is also a variable.

T : How to denote the variables? What symbols do we use to represent variables ?

S : By lower case English letters like x, y, z, u, v, etc.

T : Good. Choose a letter to denote the variable namely the length of a side of square.

S : Let x denote the length of side of a square.

T : Good! What will be its perimeter?

S : $x + x + x + x = 4x$ units.

T : What will be its area?

S : Area of a square = length × length = $x \times x = x^2$ sq. units.

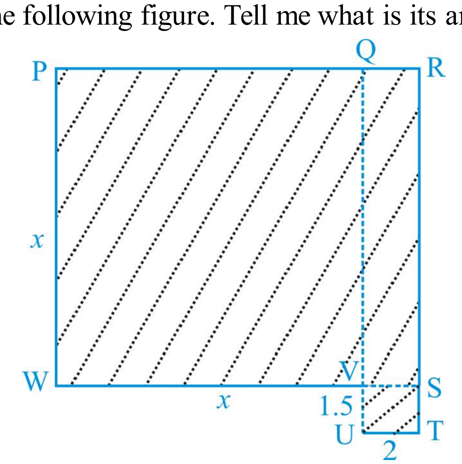
.T : What will be the volume of a cube whose side is x?

S : $x \times x \times x = x^3$ cube units.

T : What will be the volume of a cuboid whose sides are y, y and 2?

S : $y \times y \times 2 = 2y^2$ cube units.

T : Good! Look at the following figure. Tell me what is its area.

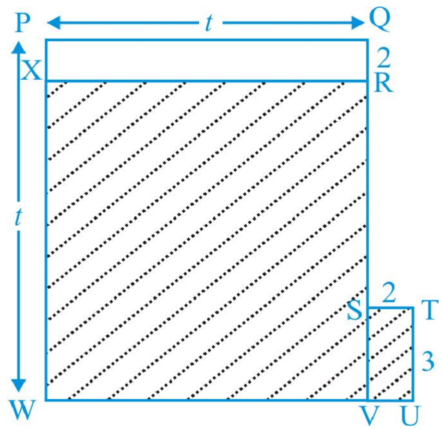


S : Area of square PQVW = x^2 sq. units

Area of rectangle QRSV = $2 \times x = 2x$ sq. units

Area of rectangle VSTU = $1.5 \times 2 = 3$ sq. units

So, the area of the figure = $(x^2 + 2x + 3)$ sq. units



T : Good! What will be the area of the shaded region in the following figure?

S : The shaded region is made up of rectangle XRVW and rectangle STUV.

T : How to get the rectangle XRVW from the square PQVW?

S : By removing the rectangle PQRX from the square PQVW.

T : So, can you tell me what is the area of the rectangle XRVW?

S : Area of rectangle XRVW = Area of square PQVW – Area of rectangle PQRX

$= (t \times t - t \times 2)$ sq. units

$= (t^2 - 2t)$ sq. units

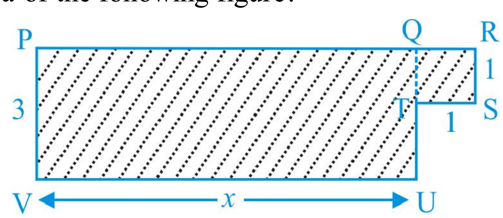
T : Good. What is the area of rectangle STUV?

S : Area of rectangle STUV = $2 \times 3 = 6$ sq. units

T : So, what is the area of the shaded region?

S : Area of the shaded region is $(t^2 - 2t + 6)$ sq. units

T : What is the area of the following figure?



S : Area of the shaded region = Area of rectangle PQUV + Area of square QRST

$= 3 \times x + 1 \times 1$

$= (3x + 1)$ sq. units

T : Can you list now the perimeters, areas and volumes which we have calculated so far?

S : $4x$, x^2, x^3, $2y^2$, $x^2 + 2x + 3$, $t^2 - 2t + 6$, $3x + 1$

T : The algebraic expressions like

$4x$, x^2, x^3, $2y^2$, $x^2 + 2x + 3$, $t^2 - 2t + 6$, $3x + 1$, etc., are called **polynomials in one variable.** Some more examples of polynomials, in one variable are

$u^3 - 2u^2 + 4u + 3$, $t^4 + 2t^3$, $z^2 - 3z$, $x^5 + 6x^2 - 3$, v^{10}

$u^3 - 2u^2 + 4u + 3$ is a polynomial in the variable u, $t^4 + 2t^3$ is a polynomial in the variable t, $z^2 - 3z$ is a polynomial in the variable z.

Tell me, what is the variable in the polynomial $v^7 - 2v^3 + 6$?

S : The variable is v.

T : Good!

In the polynomial $u^3 - 2u^2 + 4u + 3$; u^3, $-2u^2$, $4u$ and 3 are called the terms of the polynomial. 3 does not contain u and hence it is called the constant term.

What are the exponents of u in different non-constant terms of $u^3 - 2u^2 + 4u + 3$?

No response!

T : What is the exponent of u in the term u^3?

S : 3

T : What is the exponent of u in the term $-2u^2$?

S : 2

T : What is the exponent of u in the term $4u$?

: No response!

T : What is u^1 ?

S : u

T : Can you now tell what is the exponent of u in $4u$?

S : 1

T : So the exponents of u in non-constant terms of $u^3 - 2u^2 + 4u + 3$ are 3, 2 and 1, respectively.

Tell me, what are the exponents of z in non-constant terms of $z^2 - 3z$?

S : 2 and 1

T : In $x^5 + 6x^2 - 3$?

S : 5 and 2

T : So, what can you say about the exponents of the variable of the non-constant terms of a polynomial?

S : They are integers.

T : Just integers? Look at their sign!

S : They are positive integers.

T : Yes, a polynomial in one variable consists of non-constant terms with positive integeral powers of the variable.

Can you give now some more examples of polynomials in one variable?

S : $x^4 - 2x^3$, $y^3 - 3y^2 + 2y + 6$, $t^7 - 4t^6 + 5t^5 - 2t^4 + 3t^3 + t^2 - t - 9$

T : Good. Now tell me, Is $x^2 + x^{-1}$ a polynomial in one variable?

S$_1$: Yes.

S$_2$: No, Sir

T : Why?

S$_2$: In $x^2 + x^{-1}$, the term x^{-1} is not a constant term, and the exponent of x in x^{-1} is -1 which is not a positive integer. So, it is not a polynomial in the variable x.

T : Is $x^3 + 3x^2 + x^{½} + 1$ a polynomial in the variable x?

S : No. The exponent of x in the non-constant term $x^{½}$ is ½, which is not a positive integer.

T : Good. The polynomial $y^3 - 3y^2 + 2y + 6$ can be written as $(1 \times y^3) + (-3) \times y^2 + (2 \times y) + 6$. Here 1 is called the coefficient of y^3, -3 is called the coefficient of y^2, 2 is called the coefficient of y.

Tell me what is the coefficient of u^2 in $u^5 - 4u^3 + u^2$?

S : The coefficient of u^2 in $u^5 - 4u^3 + u^2$ is 1.

T : What is the coefficient of u^3?

S$_1$: 4

S$_2$: No Sir! it is – 4.

T : Good

T : What is the coefficient of x^3 in $x^5 + 4x^4 - 2x + 7$?

S : There is no x^3 term.

T : What is $0 \times x^3$?

S : 0

T : Tell me now, what is the coefficient of x^3 in $x^5 + 4x^4 - 2x + 7$?

S_1 : Sir, there is no term containing x^3. So, we cannot say anything about coefficient of x^3.

S_2 : 0

T : Why?

S : We can write

$x^5 + 4x^4 - 2x + 7$ as $x^5 + 4x^4 + (0 \times x^3) - 2x + 7$.

So, the coefficient of x^3 is 0.

Review Questions

1. Give an example of a polynomial in the variable x consisting of 3 terms.
2. Give an example of a polynomial in the variable t consisting of only one term.
3. Give an example of a polynomial in the variable y in which coefficient of y is zero.
4. What are the powers of x in the non-constant terms of the polynomial $2x^5 - 4x^3 + x^2 + 3$?
5. What is the constant term in the polynomial $y^3 - 2y$?
6. Is $x^2 - 2x^{1/3}$ a polynomial in x? Why?
7. Is $t^3 - 4t^{-3}$ a polynomial in t? Why?

Assignment for the Teachers

Give an alternate set of situations through which you will introduce polynomials to your students.

(ii) Zeroes of a Polynomial

Students often have difficulty in finding the value of a polynomial corresponding to the given value of the variable. This is mainly because they have not understood that $3x^2$, for example, means $3 \times x^2$. We tell our students that the value of a polynomial is obtained by substituting the value of the variable in the given polynomial. So, to find the value of $3x^2$ when $x = 2$, they blindly write it as 32^2 rather than 3×2^2. Similarly, for finding the value of $2x^2 + 3x - 1$ at $x = 3$, blindly they are likely to write $23^2 + 33 - 1$ rather than $(2 \times 3^2) + (3 \times 3) - 1$. So it is very necessary that when we introduce polynomials, we make them understand that $2x^2 + 3x - 1$ actually stands for $(2 \times x \times x) + (3 \times x) - 1$. Practising this with the students for various polynomials will enable them to calculate the value of a polynomial correctly for a given value of the variable. So, whether the students are able to find the value of a polynomial for a given value of the variable correctly or not has to be ascertained by the teacher before introducing the zeroes of a polynomial.

T : Consider the polynomial $p(x) = 2x^3 + 5$. What value do you get if you replace x by 1 in $p(x)$?

S : $p(1) = (2 \times 1^3) + 5 = (2 \times 1) + 5 = 7$

T : By replacing x by 1 in $p(x) = 2x^3 + 5$, we get the value 7. This number 7 is called the value of $p(x) = 2x^3 + 5$ at $x = 1$.

What value do we get if we replace x by –1 in $p(x) = 2x^3 + 5$?

S : $p(-1) = 2 \times (-1)^3 + 5 = -2 + 5 = 3$

The value of $p(x) = 2x^3 + 5$ at $x = -1$ is 3.

T : Good. What is the value of $q(x) = x^3 - 2x^2 + 3x - 6$ at $x = 0$?

S : $q(0) = 0^3 - (2 \times 0^2) + (3 \times 0) - 6 = -6$

Value of $q(x)$ at $x = 0$ is –6.

T : What is the value of $q(x)$ at $x = 1$?

S : $q(1) = 1^3 - (2 \times 1^2) + (3 \times 1) - 6 = -4$

Value of $q(x)$ at $x = 1$ is – 4.

T : What is its value at $x = -1$?

S : $q(-1) = (-1)^3 - 2 \times (-1)^2 + 3 \times (-1) - 6 = -12$

Value of $q(x)$ at $x = -1$ is –12.

(Encourage the students to give answers in full sentences.)

T : What is the value of $q(x)$ at $x = 2$?

S : $q(2) = 2^3 - (2 \times 2^2) + (3 \times 2) - 6 = 0$

Value of $q(x)$ at $x = 2$ is 0.

T : You have seen that the values of $q(x)$ at $x = 0, 1, -1$ are not zero (0). But its value at $x = 2$ is zero (0). Putting the value 2 for x in $q(x)$ makes the value of $q(x)$ as zero. So, we call 2 as a **zero** of $q(x)$.

Now find the value of $p(x) = x^2 - 4x + 3$ at $x = 1$.

S : $p(1) = 1^2 - (4 \times 1) + 3 = 0$

The value of $p(x)$ at $x = 1$ is 0.

T : So what can you say about '1' for the polynomial $p(x)$?

S : 1 is the value of x which makes the value of $p(x)$ as 0.

1 is a zero of the polynomial $p(x) = x^2 - 4x + 3$

T : What is the value of $p(x) = x^2 - 4x + 3$ at $x = 3$?

S : $p(3) = 3^2 - (4 \times 3) + 3 = 0$. Value of $p(x) = x^2 - 4x + 3$ at $x = 3$ is also zero (0). Thus, 3 is also a zero of the polynomial $p(x)$.

T : The values of $p(x) = x^2 - 4x + 3$ for $x = 1$ and $x = 3$, namely $p(1)$ and $p(3)$, are 0 and we have called 1 and 3 as zeroes of $p(x) = x^2 - 4x + 3$.

So, can you tell me when is a number c a zero of a polynomial $p(x)$?

S : c is zero of a polynomial $p(x)$ if value of $p(x)$ at $x = c$ is zero, i.e., if $p(c) = 0$.

T : Yes, if for a polynomial $q(x)$, $q(a) = 0$ then 'a' is a zero of the polynomial $q(x)$.

Now, tell me, is -1 a zero of the polynomial $x^2 + 4x + 3$?

S : Value of $x^2 + 4x + 3$ at $x = -1$ is

$(-1)^2 + (4 \times (-1)) + 3 = 1 - 4 + 3 = 0$

So, -1 is a zero of $x^2 + 4x + 3$.

T : Is 1 a zero of $x^2 + 4x + 3$?

S : $(1)^2 + (4 \times 1) + 3 = 1 + 4 + 3 = 7 \neq 0$

So, 1 is not a zero of $x^2 + 4x + 3$.

T : Is -3 a zero of $x^2 + 4x + 3$?

S : $(-3)^2 + (4 \times (-3)) + 3 = 9 - 12 + 3 = 0$

So, -3 is a zero of $x^2 + 4x + 3$.

T : Is 2 a zero of $x^2 - 2x + 3$?

S : $(2)^2 - (2 \times 2) + 3 = 4 - 4 + 3 = 3 \neq 0$

So, 2 is not a zero of $x^2 - 2x + 3$.

T : Is -2 a zero of $y^3 + 2y^2 - 2y - 4$?

S : $(-2)^3 + (2 \times (-2)^2) - 2(-2) - 4 = -8 + 8 + 4 - 4 = 0$

So, -2 is a zero of $y^3 + 2y^2 - 2y - 4$.

T : Good. Check whether the given values of the variables are the zeroes of the corresponding polynomials :

1. $x = 1$, $x^3 - 2x + 1$
2. $y = 0$, $y^2 - 2y + 1$
3. $y = -2$, $y^3 + 2y^2 - y - 4$
4. $t = 3$, $t^2 - 2t - 3$
5. $t = 1$, $t^2 + 2t - 1$
6. $v = 0$, $v^3 - 3v^2 - 4v$

Students solve the above problems in groups in a cooperative learning set up, teacher facilitating if required and encouraging the students to discuss the answers in their own way.

T : Can you guess when $x = 0$ is a zero of a polynomial $p(x)$?.

Think!

No response!

T : What happens when we substitute 0 for the variable in all the terms of the given polynomial?

S : All the terms become equal to zero (0).

T : Are you sure?

S : Constant term does not become zero. It will remain as it is.

T : So, what is the value of a polynomial when we substitute zero for the variable?

S : The value of the polynomial is equal to the constant term of the polynomial.

T : But for 0 to be a zero of a polynomial, the value of the polynomial must be zero. So can 0 be a zero of the given polynomial, if the constant term of the polynomial is not zero?

S : 0 is a zero of the given polynomial only if its constant term is equal to 0.

T : Very good.

Students have often problem to understand that a non-zero number can be a 'zero' of a polynomial. It needs a lot of illustrative examples and effective explanation on the part of the teacher for the student to understand that a non-zero number can be a 'zero' of a polynomial.

Assignment for the Teachers

Write an alternate teacher-pupil transaction for learning zeroes of a polynomial by the students.

(iii) Solution of Quadratic Equations by the Method of Completing the Square

T : Consider the equation $x^2 - 4 = 0$. What type of equation is this?

S : This is a quadratic equation in the variable x.

T : Anything special about this quadratic equation?

S : First degree term in x is not present.

T : Let us solve $x^2 - 4 = 0$,

i.e., what is the value of x for which $x^2 - 4 = 0$?

By adding 4 on both sides of $x^2 - 4 = 0$, what do we get ?

S : $x^2 = 4$

T : For what value of x is $x^2 = 4$?

S : $x = 2$

T : Good. Is there any other value of x for which $x^2 = 4$?

S : $x = -2$

T : $(2)^2 = 4$, $(-2)^2 = 4$. What are 2 and -2 for the number 4?

S : They are the square roots of 4.

T : So how to find the values of x for which $x^2 = 4$?

S : By taking the square roots of 4.

T : Good. Can you tell me now, how did we solve $x^2 – 4 = 0$?

S : To solve $x^2 – 4 = 0$, we have added 4 on both sides to get $x^2 = 4$ and then took the square roots.

T : Yes. Suppose instead of $x^2 – 4 = 0$, we had considered $x^2 – 7 = 0$, can you solve it?

S : Yes. Adding 7 on both sides, we get $x^2 = 7$. So, $x = \pm \sqrt{7}$.

T : Good. Now solve the equation $x^2 + 4 = 0$.

S : Adding – 4 on both sides, we get $x^2 = – 4$

T : So, what is the value of x?

S : We cannot find x.

T : Why?

S : There is no real number whose square is – 4.

T : Good. Solve the equation $x^2 + k = 0$.

S : Adding – k on both sides, $x^2 = – k$.

T : If k is positive, then – k is negative and there is no real number whose square is – k. So, there is no value of x for which $x^2 + k = 0$. Thus,

if k is positive, the equation $x^2 + k = 0$ has no solution.

S : What happens if k is negative?

T : If k is negative, – k is positive and so we can find the square roots of – k. The solutions of $x^2 + k = 0$ are $\pm \sqrt{(-\ k)}$.

Consider now the equation $(x – (3/2))^2 – 9 = 0$. Solve this equation.

S : By adding +9 on both sides, we get $(x – (3/2))^2 = 9$

By taking square roots, $x – (3/2) = \pm 3$

T : But we want value for x.

S : $x = \pm 3 + (3/2)$, i.e., $3 + (3/2)$ or $–3 + (3/2)$,

i.e., $(9/2)$ or $– (3/2)$

So, solutions of $(x – (3/2))^2 – 9 = 0$ are $(9/2)$ and $– (3/2)$.

T : Good. Let another student solve $(x + 5)^2 – 2 = 0$.

S : Adding 2 on both sides, $(x + 5)^2 = 2$.

Taking square roots, $x + 5 = \pm \sqrt{2}$

So, $x = -5 \pm \sqrt{2}$, i.e., $x = -5 + \sqrt{2}$ or $-5 - \sqrt{2}$

T : Look at the equation $(x - (3/2))^2 - 9 = 0$ once again.

Can you expand the L.H.S. and simplify it ?

S : $(x - (3/2))^2 - 9 = x^2 - 2 \times x \times (3/2) + (-3/2)^2 - 9$

$= x^2 - 3x + (9/4) - 9$

$= x^2 - 3x - (27/4)$

T : So, the equation $(x - (3/2))^2 - 9 = 0$ is the same as $x^2 - 3x - (27/4) = 0$.

So, what can you say about the solution of $x^2 - 3x - (27/4) = 0$?

S : The solution of $x^2 - 3x - (27/4) = 0$ is the same as the solution of $(x - (3/2))^2 - 9 = 0$.

So, the solutions of $x^2 - 3x - (27/4) = 0$ are $9/2$ and $-(3/2)$.

T : Similarly, expand the square and simplify $(x + 5)^2 - 2 = 0$.

S : $x^2 + 10x + 25 - 2 = 0$, i.e., $x^2 + 10x + 23 = 0$.

T : So, what are the solutions of $x^2 + 10x + 23 = 0$?

S : The solutions of $x^2 + 10x + 23 = 0$ are $-5 + \sqrt{2}$ and $-5 - \sqrt{2}$.

T : So, if we write $x^2 - 3x - (27/4) = 0$ in the form $(x - (3/2))^2 - 9 = 0$, we can solve it.

Similarly, if we write $x^2 + 10x + 23 = 0$ in the form $(x + 5)^2 - 2 = 0$, we can solve it.

But, how to write $x^2 - 3x - (27/4) = 0$ in the form $(x - (3/2))^2 - 9 = 0$?

What difference do you see in the two equations?

S : In $x^2 - 3x - (27/4) = 0$, the x term $-3x$ is present. But in $(x - (3/2))^2 - 9 = 0$, the x term is not present.

T : $(x - (3/2))^2 - 9 = 0$ is the same as $x^2 - 3x - (27/4) = 0$. So, $-3x$ is also present in $(x - (3/2))^2 - 9 = 0$, but it is not seen. It has been included in $(x - (3/2))^2$.

In fact,

$$
\begin{aligned}
x^2 - 3x &= x^2 + (2 \times x \times (-3/2)) \\
&= x^2 + (2 \times x \times (-3/2)) + (-3/2)^2 - (-3/2)^2 \\
&= (x - (3/2))^2 - (-3/2)^2
\end{aligned}
$$

Thus, $-3x$ is taken inside the whole square $(x - (3/2))^2$ by adding and subtracting $(-3/2)^2$.

So,

$$x^2 - 3x - (27/4) = (x - (3/2))^2 - (-3/2)^2 - (27/4)$$
$$= (x - (3/2))^2 - (9/4) - (27/4)$$
$$= (x - (3/2))^2 - 9$$

Thus, $x^2 - 3x - (27/4) = 0$ is converted into $(x - (3/2))^2 - 9 = 0$ by adding and subtracting $(-3/2)^2$. The method of writing $x^2 - 3x - (27/4) = 0$ as $(x - (3/2))^2 - 9 = 0$ by adding and subtracting $(-3/2)^2$ is known as ***the method of completing the square.***

Do this now for $x^2 + 10x + 23 = 0$.

S : $$x^2 + 10x = x^2 + (2 \times x \times 5)$$
$$= x^2 + (2 \times x \times 5) + 5^2 - 5^2$$
$$= (x + 5)^2 - 25$$

So, $x^2 + 10x + 23 = (x + 5)^2 - 25 + 23 = (x + 5)^2 - 2$.

T : So, $10x$ has been brought inside $(x + 5)^2$ by adding and subtracting 5^2 and we have written $x^2 + 10x + 23 = 0$ as $(x + 5)^2 - 2 = 0$.

Thus, the equation $x^2 + 10x + 23 = 0$ is solved by converting it as $(x + 5)^2 - 2 = 0$ by the method of completing the square and then solving $(x + 5)^2 - 2 = 0$.

T : Solve $x^2 + 6x - 7 = 0$ by the method of completing the square.

S : First we should consider $x^2 + 6x$ and rewrite it as

$$x^2 + 6x = x^2 + (2 \times x \times 3)$$
$$= x^2 + (2 \times x \times 3) + 3^2 - 3^2$$
$$= (x + 3)^2 - 9$$

So, $$x^2 + 6x - 7 = (x + 3)^2 - 9 - 7$$
$$= (x + 3)^2 - 16$$

T : Good, you have added and subtracted 3^2. But why have you added and subtracted 3^2?

S : Because $6x = 2 \times x \times \underline{3}$

That is why I have added and subtracted 3^2.

T : Good. Proceed now with solving of the equation $x^2 + 6x - 7 = 0$.

S : Solving $x^2 + 6x - 7 = 0$ is the same as solving $(x + 3)^2 - 16 = 0$.

$(x + 3)^2 = 16$ by adding 16 on both sides of $(x + 3)^2 - 16 = 0$.

So, $x + 3 = \pm 4$

That is, $x = 1$ or -7.

The solutions of $x^2 + 6x - 7 = 0$ are 1 and -7.

T : Good. Solve now $x^2 - 3x + 4 = 0$ by the method of completing the square.

S : $x^2 - 3x = x^2 + (2 \times x \times (-3/2))$

$= x^2 + (2 \times x \times (-3/2)) + (-3/2)^2 - (-3/2)^2$

$= (x - (3/2))^2 - 9/4$

So, $x^2 - 3x + 4 = (x - (3/2))^2 - 9/4 + 4 = (x - (3/2))^2 + 7/4$

So, $x^2 - 3x + 4 = 0$ is the same as $(x - (3/2))^2 + 7/4 = 0$.

T : Solve now for x.

S : $(x - (3/2))^2 + 7/4 = 0$

So, $(x - (3/2))^2 = - 7/4$

But we cannot find square root.

T : Why?

S : Because there is no real number whose square is $(-7/4)$.

T : So, what is the solution for $(x - (3/2))^2 + 7/4 = 0$?

S : There is no solution for $(x - (3/2))^2 + 7/4 = 0$ in real numbers.

T : So, what is the solution for $x^2 - 3x + 4 = 0$?

S : There is no solution.

T : Why?

S : Because, the solution of $x^2 - 3x + 4 = 0$ is the same as that of $(x - (3/2))^2 + 7/4 = 0$ and $(x - (3/2))^2 + 7/4 = 0$ has no solution in real numbers.

T : Thus, by the method of completing the square, you can find out if there is solution for the given quadratic equation or not. If it has solutions, you can find them.

Review Questions

By the method of completing the square, solve the following quadratic equations:

1. $x^2 - 5x + 6 = 0$
2. $x^2 + 2x + 3 = 0$
3. $y^2 - (2/3)y - (8/9) = 0$
4. $3x^2 - 4x + 5 = 0$
5. $t^2 - t + 2 = 0$

These problems can be worked in groups in cooperative learning set up. Encourage each student to participate and also discuss among themselves what they have found, raise questions, discuss the responses and give reasons for their responses.

T : Let us now see how the method of completing the square can be applied to solve some problems. Consider the following problem:

A train travels 360 km at a uniform speed. If the speed had been 5 km/h more, the train would have taken one hour less for the same journey. What is the original speed of the train?

Read the problem and tell me, what you are asked to find out.

S : The original speed of the train.

T : So, what is the unknown?

S : The original speed of the train.

T : Use a symbol to represent the original speed of the train.

S : Let the speed of the train be x km/h. (Encourage the students to respond in full sentences).

T : What is known? What is given in the problem?

S : Train travels 360 km.

T : So, what is the distance travelled by the train?

S : Distance travelled = 360 km.

T : You are also given some more information about the time taken by the train. If the distance travelled is 360 km and the speed of the train is x km/h, what will be the time taken by the train for the journey?

S : Time taken = $\dfrac{\text{Distance travelled}}{\text{Speed of the train}} = \dfrac{360}{x}$ hours

T : Good. You are also given some more information about the time taken by the train. Read the problem once more and tell me what it is.

S : If the speed had been 5 km/h more, the train would have taken 1 hour less to travel.

T : If the speed of the train is 5 km/h more, what will be its new speed?

S : $(x + 5)$ km/h

T : Why?

S : Original speed was x km/h

T : Good. What will be the time taken for the journey with the new speed ?

S : New time taken = $\dfrac{\text{Distance travelled}}{\text{New speed of the train}} = \dfrac{360}{x+5}$ hours .

T : Time taken by the train with original speed was $360/x$ hours and time taken by the train with the new speed = $360/(x + 5)$ hours.

What is given in the problem about the time taken with the original speed and the new speed?

S : Time taken by the train with the new speed is one hour less than the time taken with the original speed.

T : Can you write it in the form of an equation?

S : $360/(x + 5) = (360/x) - 1$, i.e., $(360/(x + 5)) - (360/x) + 1 = 0$.

T : But the unknown x is in the denominator. Can you bring it to the numerator?

S : Yes, by multiplying through out by $(x + 5)\, x$.

T : Do it.

S : $360x - 360(x + 5) + (x + 5)x = 0$

i.e., $360x - 360x - 360 \times 5 + x^2 + 5x = 0$

i.e., $-1800 + x^2 + 5x = 0$

i.e., $x^2 + 5x - 1800 = 0$

T : So what equation does the original speed x km/h of the train satisfy?

S : x satisfies the quadratic equation $x^2 + 5x - 1800 = 0$.

T : We have to find the original speed x of the train. To find x, you have to solve this equation for x.

So, solve $x^2 + 5x - 1800 = 0$ for x.

S : $x^2 + 5x = x^2 + (2 \times x) \times (5/2)$

$= x^2 + (2 \times x) \times (5/2) + (5/2)^2 - (5/2)^2$

$= (x + (5/2))^2 - 25/4$

So, $x^2 + 5x - 1800 = (x + (5/2))^2 - 25/4 - 1800 = (x + (5/2))^2 - 7225/4$

So, the equation $x^2 + 5x - 1800 = 0$ is the same as $(x + (5/2))^2 - 7225/4 = 0$

So, $x + (5/2) = \pm\sqrt{7225/4}$

$= \pm\left(\sqrt{7225}\right)/2$

T : You aheady know how to find square root of a number. Find it.

S : 5 | 7225

5 | 1445

17 | 289

17

$\sqrt{7225} = \sqrt{5 \times 5 \times 17 \times 17} \leftarrow$ Prime factorisation

So, $\sqrt{7225} = \sqrt{5 \times 5 \times 17 \times 17} = 5 \times 17 = 85$

So, $x + (5/2) = \pm(85/2)$

Therefore, $x = -(5/2) + (85/2)$ or $-(5/2) - (85/2)$

$= 40$ or -45

T : But what is x? Can it be negative?

S : x is the original speed of the train, therefore, it cannot be negative.

So, $x = 40$ km/h

T : Good. So you have found the original speed of the train as 40 km/h.

Review Question

The diagonal of a rectangular field is 60 metres more than the shorter side. If the longer side is 30 metres more than the shorter side, find the sides of the field.

Assignment for the Teacher:

Write the student-teacher interaction for making the students solve the above problem on rectangular field.

(iv) Pair of Linear Equations in Two Variables

T : In your earlier classes, you have learnt linear equations in one variable. Can you give a few examples?

S : $3x + 5 = 0$, $y - 2 = 5$, $2x - 3 = 1$

T : Why are they called linear equations in one variable?

S : Each equation has exactly one variable.

T : Yes, but why are these called linear equations?

S : Because the degree of non-constant term in each equation is 1.

T : So, can you say what is a linear equation in one variable?

S : An equation in which there is exactly one variable and the degree of each non-constant term in the equation is 1.

T : Good. Consider the following problem:

Ram and Rahim together have 25 books. Rahim has 5 more books than that of Ram. Find the number of books Ram and Rahim individually have.

S : Let the number of books with Ram be x.

Then the number of books with Rahim $= x + 5$

Ram and Rahim together have 25 books

So, $x + (x + 5) = 25$

Therefore, $2x + 5 = 25$

i.e., $2x = 25 - 5$

$= 20$

So, $x = 10$ and $x + 5 = 15$.

Therefore, Ram has 10 books and Rahim has 15 books.

T : Good. Let us consider another problem.

Gita went to a shop for buying some toilet articles. She found that there were two combo offers. One combo pack of 2 buckets and 3 mugs was costing ` 225 and another combo pack of 3 buckets and 2 mugs was costing ` 315. Can you find the cost of each bucket and each mug?

What are you asked to find here?

S : The cost of one bucket and cost of one mug.

T : You do not know them. So, what are the unknowns?

S : The unknowns are cost of one bucket and cost of one mug.

T : Represent the two unknowns by some symbols.

S : Let the cost of a bucket be ` $x$ and cost of a mug be ` y.

T : What is given in the problem?

S : Cost of 2 buckets and 3 mugs is ` 225.

Cost of 3 buckets and 2 mugs is ` 315.

T : If the cost of a bucket is ` $x$ and the cost of a mug is ` y, what will be the cost of 2 buckets and 3 mugs?

S : ` $2x$ + ` $3x$ = ` $(2x + 3y)$.

T : Good. What will be the total cost of 3buckets and 2 mugs?

S : ` $(3x + 2y)$.

T : So, write what is given in the problem in the equation form.

S : $2x + 3y = 225$, and

$3x + 2y = 315$

T : How many variables are there in each of these equations?

S : There are two variables in each of the equations.

T : What is the degree of each non-constant term in the above equations?

S : Each non-constant term has degree 1.

T : Such equations are called linear equations in two variables.

$2u - v = 0$, $x + 3z - 5 = 6$, $3r - t = 2$ are some more examples of linear equations in two variables. Give some more examples.

S : $x + 2y = 3$

$u - 3v = 4$

$3t - 5u = 8$

T : Is $u = 2v + 5$ a linear equation in two variables?

S : Yes.

T : Good.

Is $x - y + z = 8$ a linear equation in two variables?

S : No.

T : Is $u - 2v^2 = -2$ a linear equation in two variables?

S : No.

T : Why?

S : There are two variables u, v in the given equation.

S : But the degree of the term $-2v^2$ is 2. So, it is not a linear equation.

T : Good. In the following equations identify which equations are linear equations and which are not. Give reasons.

1. $x - 2z = 3$
2. $2r - 5s = 0$
3. $2x - 3t = 5t + 1$
4. $r - s + 2u = 7$
5. $z + 5 = 3z$
6. $2u + 3v - z = 0$
7. $2x + y + t + 3 = 5$
8. $2x + 3t = 2t + 3$
9. $u - 2x^2 = 3$
10. $2u^3 - 4v = 1$
11. $u^2 - t^2 = 4t$
12. $x - y = x + 3y + 4$

This can be carried out in groups in cooperative learning set up as told earlier.

T : We had considered earlier a problem situation of Gita going to a shop for buying buckets and mugs. We will consider one more situation now.

Each student in Class IX has 4 textbooks and 6 notebooks. In Class III, each student has 2 textbooks and 2 notebooks. In Class IX and Class III put together total number of textbooks is 100 and total number of notebooks is 140. How to represent this information in the form of equations ? (Pause)

T : Do we know the number of students in Class IX and the number of students in Class III?

S : No.

T : So what are the unknowns?

S : Number of students in Class IX is one unknown and number of students in Class III is another unknown.

T : Represent these unknowns by symbols and tell me how many textbooks in all will be there in Class IX and how many textbooks in all in Class III?

S : Let x denote the number of students in Class IX and y denote the number of students in Class III. Each student in Class IX has 4 textbooks. So, the number of textbooks in Class IX is $4x$.

T : Good, another student tell me how many textbooks are in all in Class III?

S : Each student in Class III has 2 textbooks and the number of students in Class III is y.

So, the number of textbooks in Class III is $2y$.

T : So, how many textbooks in all together in Class IX and Class III?

S : $4x + 2y$

T : But in the problem what is given about textbooks in Class IX and Class III put together and can we write this information in the form of an equation?

S : The total number of textbooks is 100. So,

$4x + 2y = 100$

T : Similarly, express the number of notebooks in Class IX and Class III together.

S : Number of notebooks in Class IX is $6x$.

Number of notebooks in Class III = $2y$.

Given that total number of notebooks in Class IX and Class III put together is 140. So, $6x + 2y$ is 140.

T : So, the number of students x in Class IX and number of students y in Class III satisfy the following linear equations in two variables:

$$4x + 2y = 100$$

$$6x + 2y = 140$$

Review Questions

Write the linear equations in two variables for the following problem situations:

1. Prabha has some ₹ 5 notes and some ₹ 10 notes. The number of ₹ 5 notes is one more than the number of ₹ 10 notes and total value of the money is ₹ 125.
2. Twice the number of girls in a class is 5 more than 3 times the number of boys in the class. The total number of students in the class is 72.

Assignment for the Teachers

Write student-teacher transaction for the learning of solution of two simultaneous linear equations by elimination method.

2.5 Misconceptions

1. At $x = 2$, the value of $5x^4$ is written as 52^4 in place of 5×2^4.
2. Zero of a polynomial is understood as 0 while zero of a polynomial can be any real number.
3. Sometimes, coefficient of a term is taken as degree of the term.
4. Negative sign of a term is not included in the numerical coefficient of the term.

   for example, in the polynomial $43x^3 - 3x^2 + 1$, coefficient of the term $-3x^2$ is taken as 3 instead of -3. Also, $3x^2$ is taken as a term in place of $-3x^2$.
5. Meaning of zero and root are considered as the same while we speak of zero in case of polynomial and root in case of polynomial equation.
6. $(x - 4)(x - 2) = 4$ implies $x - 4 = 4$, $x - 2 = 4$ as they do in case of

   $(x - 4)(x - 2) = 0$, $x - 4 = 0$, $x - 2 = 0$.
7. In solving $x^2 = -4$, some students find $x = \pm\sqrt{-2}$.
8. Sometimes they write remainder theorem for factor theorem and vice-versa.

UNIT
3

TEACHING OF GEOMETRY

3.1 Introduction

The word ‘geometry’ is believed to have been derived from two Greek words ‘geo’ meaning ‘earth (land)’ and ‘metron’ meaning ‘to measure’. Thus, the origin of geometry can be traced back to the period when human being first felt the need of measuring lands. Ancient Egyptians were perhaps the first people to study geometry in the process of restoring their landmarks after the annual flood of river *Nile*. They were mainly concerned with finding perimeters and areas of figures such as rectangle, square and so on. Ancient Babylonians also used geometry for finding areas of rectilinear figures and developed a number of formulae for areas of these figures. These formulae are available in ancient Babylonian’s mathematical text Rhind Papyrus (1650BC). Both Egyptians and Babylonians used geometry for practical purposes, namely for measuring lands and constructing buildings. Ancient Indians also used geometry only for practical purposes, namely for construction of different types of *altars (Vedies)* for performing religious rituals and also in astronomy and astrology. The necessary measurements for constructing altars were done with the help of a ‘rope’ called ‘*sulba*’. This information regarding the geometrical knowledge of the vedic seers was contained in ancient texts known as *sulbasutras* composed during the period 800 BC to 500 BC. The oldest of the known sulbasutras, namely the Baudhayana Sulbasutras (about 800 BC), contains a clear statement of the so-called Pythagoras Theorem in the form “The diagonal of a rectangle produced by itself both (the areas) produced separately by its two sides.” Thus, most of the ancient civilisations used geometry only for practical purposes and very little was done by them to make it a systematic study.

The credit for giving geometry a systematic treatment goes to Greeks. In this connection, a special mention can be made of 'Thales' because it was due to him the knowledge of geometry passed on from Egyptians to Greeks. The most famous pupil of Thales was Pythagoras (580 BC-500 BC). The best known of the Greek mathematicians is Euclid (who lived around 300 BC). He initiated a new process of thinking called logical reasoning and arriving at conclusions about some geometric figures such as rectangles, triangles, etc, based on points, lines, etc., and certain axioms and postulates.

3.2 General Strategies

There are two ways of approaching the subject- one is the logical approach and the other is the psychological approach. In the first approach, learning involves the systematic deductions of the facts through a sequence of logical steps, while in the second approach, learning is based on the need, curiosity and interest of the learner. For the second approach, therefore, it is suggested that geometry should start from solids and gradually come to surfaces (planes), lines and points and only then to abstract concepts. You might have seen that this approach is followed up to the primary stage of schooling. The first approach of teaching geometry is followed at the secondary stage, which is for the present in our consideration. It might be interesting to note for some of you that a mid-way approach is followed for the teaching of geometry for upper primary stage, which may be termed as a semi-logical approach or an experimental approach. In other words, a systematic study of the subject has already started in an informal manner from the upper primary stage and it is consolidated at the secondary stage. The essential constituents of this systematic study are as follows:

(a) Undefined Terms

(b) Definitions or Defined Terms

(c) Axioms or Postulates

(d) Theorems

(e) Riders and Constructions

Beside the above, the following strategies are also helpful in teaching of geometry:

1. Inductive-deductive approaches
2. Explaining through examples
3. Asking 'why' and 'how' questions/seeking reasons for the responses
4. Developing alternate approaches for solving problems.

3.3 Key Concepts

- Basic Geometrical Concepts
- Pairs of Angles

- Transversal and Parallel Lines
- Angle Sum Property for a Polygon.
- Congruency of Figures- Different Criteria for Congruency of Triangles
- Properties of Isosceles Triangles
- Inequalities in a Triangle
- Similarity of Figures:- Different Criteria for Similarity of Triangles
- Pythagoras Theorem
- Special Quadrilaterals and their Properties
- Circles and their Properties
- Constructions.

3.4 Teaching Strategies

Now we shall discuss a few key concepts listed below for teaching-learning in a classroom situations:

(i) Basic Geometrical Concepts (ii) Congruency of Figures

(iii) Inequalities in a Triangle (iv) Similarity of Figures

(v) Pythagoras Theorem (vi) Circle

(i) Basic Geometrical Concepts

Point, line and plane are the building blocks of geometry. Teacher can explain these ideas through some physical examples and experiences as given in the textbooks. However, through some activities such as one given below, it may be made clear to the students that everything cannot be defined and for this reason, *point*, *line* and *plane* are considered as undefined terms:

Activity: Each student of the class may be asked to write the possible definition of one of these terms say 'a line' on her notebook. In each of these definitions, it will be observed that there are many 'words' which have been used to define a 'line' and these words themselves need to be defined first. You may now ask the student concerned to define these words used by her. On getting her answers, it will be seen that for defining 'these words' also some more 'new words' have been used, which are required to be defined themselves again. In this way, this process may never end and hence need of taking the line as an undefined term. Same process can be followed for explaining the need of taking point and plane as the undefined terms.

After the undefined terms, other basic geometrical concepts like collinear points, non-collinear points, intersecting, parallel and concurrent lines may be explained through these undefined terms. After these, more basic concepts like line segment, ray and angles may be taken up. In this way, new geometrical terms are defined with the help of the undefined terms (point, line and plane). After getting new definitions, attempt is then to be made to establish relationship among these new defined terms and undefined terms with the help of physical experiences. Examples of such relationships are as follows:

(a) Through two given points in a plane, one and only one line can be drawn. This line wholly lies in the plane.

(b) Two lines in a plane cannot have more than one point in common.

(c) Given a line and a point not lying on it, one and only one parallel line can be drawn to the given line through the given point (known as Playfair's Axiom), and so on.

Here, it may be emphasised that just as it is not possible to define each and every term, in the same way, it is not possible to establish each and every relationship of the above type logically, though they appear to be self evident truths. Such relationships (statements) are assumed or accepted as true without any logical reasoning and are termed as **axioms** or **postulates**. For example, out of the above three statements (a), (b) and (c), statements (a) and (c) appear to be true but it is not possible to establish their truth logically. So, they are accepted as self evident truths and called **axioms.** So far as statement (b) is concerned, its truth can be logically established. Such statements are usually referred to as **'Theorems'**. They can be directly used, along with other axioms or postulates, to establish the truth of the other statements. The process of establishing the truth of a statement in a logical manner is called a **'proof'** of the statement. These constituents of geometry have been discussed with this section of 'Basic Geometrical Concepts' because they will permeate throughout in all the topics of geometry. Therefore, it is important for all the students to be well aware with these constituents. The students must also be aware with the fact that though we are dealing with the geometry named as 'Euclidean Geometry', but axioms or postulates used by us are slightly different from those used by Euclid. For example, these days, we use the words axioms and postulates interchangeably, while Euclid used the 'postulates' for self evident truths in geometry. Students must also be aware with the fact that Euclid has defined point, line and plane (surface) in his own ways as already given in Class IX Mathematics Textbook of NCERT. It may also be emphasised by the teacher before the students that the historical 'Euclid's fifth postulate' has taken the form of present 'Playfair's Axiom'.

Review Questions

1. Discuss in what way present day study of geometry is different from the approach followed by Euclid.
2. What do you mean by proving a statement in geometry?

(ii) Congruency of Figures

Students are always in the habit of comparing objects. This natural tendency of students

may be utilized by the teacher in developing the habit of systematic comparison leading to the concepts of **congruency.** This concept may also be introduced by showing objects of the same shape and same size to the students such as **'blades'** of the same brand, **biscuits** of the same trade mark and so on. They may also be asked to think some more such objects, such as two toys made to the same details in a factory, maps of the world made to the same size and so on. Through such objects, it may be emphasised that **two figures of the same shape and same size are called congruent figures.**

The following two points need to be emphasised in the case of congruency of figures:

(i) Congruency of figures also exists in the case of three dimensional figures.

(ii) In the case of two plane figures, if one figure can be placed over the other figure so that they cover each other exactly, then they are congruent to each other.

Students must be convinced by the teacher that there can be congruent squares, congruent rectangles, congruent circles, congruent rhombuses, congruent parallelograms, etc. also.

They should also be made to understand that various criteria for congruency of triangles are akin to those for their constructions. The teacher may also emphasise that teaching of congruency of triangles and its different criteria are contained in its applications to the study of other geometric figures such as parallelograms, rectangles, circles, etc. It would be better, if the same is discussed in the class in a dialogue mode as follows:

T : You have already learnt the congruency of two triangles in earlier classes. Can you say when the two triangles will be congruent?

S_1 : If all the sides and all the angles of one triangle are respectively equal to all the sides and all the angles of the other triangle.

T : It means that all the six elements of one triangle are equal to all the six elements of the other triangle.

S_1 : Yes, Madam

T : Do you remember, how the two congruent triangles are written in the symbolic form?

S_2 : Yes, Madam

T : Suppose in two triangles ABC and PQR, AB = PQ, BC = QR, CA = RP, $\angle A = \angle P$, $\angle B = \angle Q$ and $\angle C = \angle R$. How will you write the congruency of the two triangles in symbolic form?

S_2 : $\Delta ABC \cong \Delta PQR$.

T : O.K. Can we write it as $\Delta BCA \cong \Delta QRP$?

S_2 : No.

T : Is the answer of S_2 correct?

S_3 : No, we can write it as $\Delta BCA \cong \Delta QRP$.

T : How?

S_3 : When $\Delta ABC \cong \Delta PQR$, $A \leftrightarrow P$, $B \leftrightarrow Q$ and $C \leftrightarrow R$. With this correspondence, writing $\Delta BCA \cong \Delta QRP$ is perfectly correct.

T : Is it correct to write $\Delta BCA \cong \Delta RQP$?

S_3 : No.

T : Why?

S_3 : Because, it will give the correspondence $B \leftrightarrow R$, $C \leftrightarrow Q$, and $A \leftrightarrow P$.

T : So, what?

S_3 : With this correspondence, $\angle C = \angle Q$, $\angle A = \angle P$, $BC = RQ$, $AB = PR$ and $CA = QP$. This equality of the corresponding parts is not the same as the given equality of corresponding parts of the two triangles as above.

T : It means that congruence of two triangles should always be written symbolically in a correct correspondence of vertices.

S : Yes, Madam.

T : You have just seen that for congruency of two triangles, all the six elements of one triangle must be equal to the six elements of the other triangle.

S_1 : Yes, Madam.

T : Then, for establishing the congruency of two triangles, should we always check the equality of all the six parts of the two triangles? Can we not do this with only few elements of the two triangles?

S_2 : No, we need not do the same. We can do the same with the equality of only **three elements** of the two triangles.

T : Good! You have learnt these things in your earlier classes. Do you remember the names of results used for this purpose?

S : They are called SSS, SAS, ASA and RHS criteria or rules for congruency of two triangles.

T : How were these criteria obtained?

S : Through activities and experiments.

T : Should we always rely on the results obtained through activities?

S : No.

T : What to do then?

S : We should try if we can obtain these criteria through logical reasoning. Something has already been done in our Class IX Mathematics Textbook.

S : Yes, in the book, discussion has started with the **SAS criterion**. This criterion has been established through an activity and accepted as an Axiom (or Postulate) known as **SAS axiom.**

S : Can we not prove it?

T : No. Besause, it is not possible to prove each and every result.

S : Can we assume the other three criteria also as axioms?

T : Yes, we can. But it is said that the number of axioms in any system should be kept as minimum.

$\mathbf{S_2}$: Then, what is to be done?

T : Other three criteria, namely ASA, SSS and RHS are proved. Refer to your Textbook of Class IX.

$\mathbf{S_1}$: All right Madam, but I am confused with the sequence of topics given in the textbook.

T : What is the problem?

$\mathbf{S_1}$: The sequence is quite different from what we have learnt in earlier classes.

$\mathbf{S_2}$: Yes, Madam.

T : What is the difference between the two sequences?

$\mathbf{S_3}$: In Class VII, it was as follows:

SSS, SAS, ASA, RHS criteria and then properties of isosceles triangles.

In Class IX, it is as follows:

SAS and ASA criteria. Properties of isosceles triangles, SSS and RHS criteria.

T : Yes, you are right.

$\mathbf{S_4}$: What is the need of changing the sequence in Class IX?

T : Because we are studying Geometry in Class IX in a systematic manner.

$\mathbf{S_1}$: What does it mean?

T : Here, we are starting with undefined terms, axioms (or postulates) and arriving at (proving) new results using the process of logical reasoning with the help of previously known axioms and theorems (results) which have been earlier proved independently of the result to be proved.

$\mathbf{S_1}$: I am still not clear. Please explain it a little more.

T : If you look at the formal proofs of ASA, SSS and RHS criteria, you will find that ASA criterion has been proved using SAS criterion. Further, SSS and RHS criteria are proved using SAS criterion as well as properties of isosceles triangles. Therefore, it is necessary to teach the properties of isosceles triangles before SSS and RHS criteria.

S_2 : Then why it was not done in Class VII?

T : There, we were not following a systematic or logical approach. In fact, at that stage, the approach was 'semi-logical.'

S_2 : Yes, Madam. Now, we have understood the need of this change. But cann't we have some more criteria for congruence of two triangles?

T : You must have seen one such criterion in the textbook, namely AAS criterion.

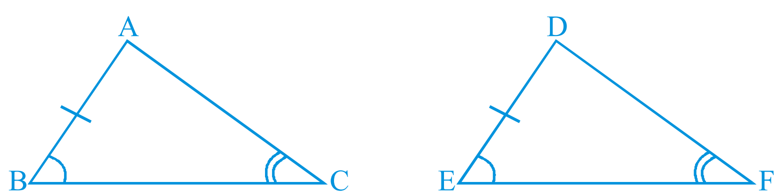
S_3 : Yes, Madam. It is as follows:

If two angles and a side of one triangle are equal to two angles and the **corresponding side** of the other triangle, then the two triangles are congruent.

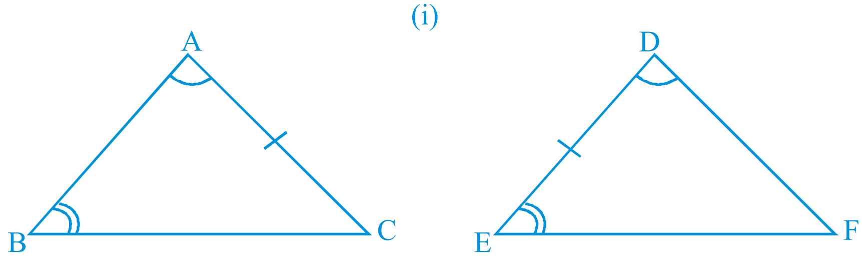
T : Very good ! Do you know that here the word **corresponding** is very important.

S_4 : Why?

T : Look at the following figures:



(i)



(ii)

In Fig.(i), $\angle B = \angle E$, $\angle C = \angle F$ and $AB = DE$. Also, AB and DE are corresponding sides, because AB is opposite to $\angle C$ and DE is opposite to $\angle F$, which is equal to $\angle C$. Hence, $\Delta ABC \cong \Delta DEF$. (AAS)

But in Fig.(ii), $\angle A = \angle D$, $\angle B = \angle E$ and $AC = DE$. But AC and DE are not corresponding sides. Hence, ΔABC is not congruent to ΔDEF.

S_1 : Really, Madam, word 'corresponding' is important. I have one more doubt in my mind. Can we have side-side-angle (SSA) a criterion?

T : No.

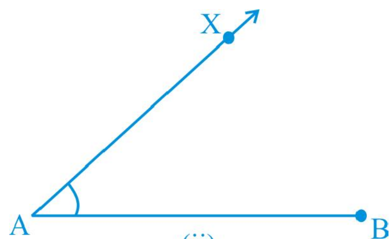
S_1 : Why not?

T : We have earlier stated that various criteria for congruency of triangles are akin to those for their constructions. Try to construct a triangle ABC in which two sides AB and BC and ∠A are given.

S_1 : First, we draw AB of given length [Fig.(i) below].

(i)

S_2 : Now, we draw an angle XAB = Given angle A [Fig.(ii) below].

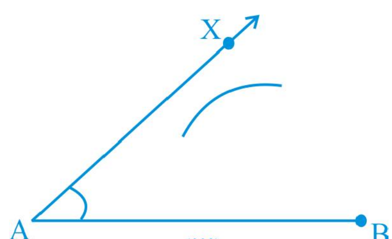


(ii)

S_3 : Now with B as centre and BC as radius, we draw an arc to obtain point C.

T : There are three posibilities:

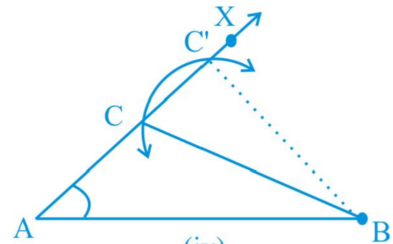
(a) This arc does not intersect arm AX of ∠XAB [Fig.(iii)]. In this case, you will not get any triangle.



(iii)

(b) This arc may intersect arm AX at two points C and C′. In such a case, there will be two triangles ABC and ABC′ with the given measurements [Fig. (iv)].

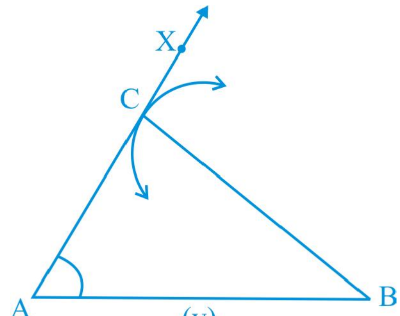
(c) This arc may intersect arm AX at a single point C [Fig. (v)]. In such a case, there will be a unique triangle ABC. Further, this triangle will be right angled at C. I think, you must have understood, why SSA cannot be taken as a criterion for congruence of two triangles.



(iv)

S$_4$ **:** Yes, Madam. But in case (c), we can say that SSA is a criterion.

T : Yes. But this case is nothing but the RHS criterion and hence it cannot be considered **separately.**



(v)

S : Thank you, Madam.

T : O.K. Tomorrow, we shall discuss something more interesting.

Next day, a scene in the classroom, Teacher enters in the class.

S : Good morning, Madam

T : Good morning ! students. We have discussed the various criteria for congruency of triangles and also about rationale of including the following two properties of the triangles in between the (SAS, ASA) and (SSS, RHS) criteria:

(i) If two sides of a triangle are equal, then the angles opposite these sides are equal and

(ii) If two angles of a triangle are equal, then the sides opposite these angles are equal.

Our next task is to apply these criteria and the above two properties in solving other geometric problems.

S : Then, how to proceed?

T : Initially, for having a better understanding of the criteria, you may attempt following type of problem:

"In triangles ABC and DEF, BC = FD and AB = EF. What extra information is needed to make the two triangles congruent using (i) SAS criterion and (ii) SSS criterion?

S_1 : Clearly for (i), the required information is $\angle B = \angle F$.

S : And for (ii), it is CA = DE.

T : Very good.

This dialogue process may also be used as a problem solving technique for proving a rider or even a theorem. For this pupose, following should be kept in mind.

(i) Enunciation of the statement of the rider or theorem should not be neglected, because it is already given in the question. It should be remembered that this provides the base, not only for that particular rider or theorem but for many riders and theorems to be learnt afterwards.

(ii) Emphasis should be on understanding rather than on writing. Even a minute point should be discussed and students should provide as many arguments as possible.

(iii) Wherever necessary, we should try to draw a rough figure, clearly explaining the given conditions.

As an example, following problem may be considered:

"Prove that every angle of an equilateral triangle is 60°"

In the dialogue form, it may be discussed as follows:

(Scene in a classroom)

T : What is an equilateral triangle?

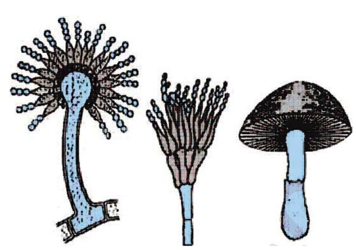
S_1 : A triangle whose two sides are equal.

T : S_2, is S_1 correct?

S_2 : No, Madam. He is talking of an isosceles triangle only.

T : What is correct, then?

S_2 : A triangle whose all three sides are equal.



Aspergillus *Penicillum* *Agaricus*

Fungi

T : Good ! If ABC is an equilateral triangle, then what can be said of sides BC, CA and AB? (You may look at the adjoining figure)

S_3 : Madam, AB = BC = CA

T : Is it correct?

S : Yes, Madam.

T : If AB = BC, what will happen to angles of the triangle ?.

S : Angles opposite these sides are equal.

T : Which two angles will be equal ?.

S : $\angle A = \angle C$

T : Very good. What about $\angle A$ and $\angle B$? Are they also equal?

S : They are also equal. I mean $\angle A = \angle B$.

T : Why?

S : Because, CB = CA

T : Good. Then, is $\angle A = \angle B = \angle C$?

S : Yes, Madam.

T : How?

S : Because $\angle A = \angle C$ and $\angle A = \angle B$.

T : OK; Do you know the value of $\angle A + \angle B + \angle C$?

S : It is equal to 180°, Madam

T : Good ! What can you say about $\angle A$?

S : $\angle A = 180° \div 3 = 60°$

T : What about $\angle B$ and $\angle C$, then?

S : They are also equal to 60° each.

T : What have you learnt or proved?

Whole class: Every angle of an equilateral triangle is of 60°.

While discussing the applications of various criteria for congruency of two triangles, teacher should also highlight and prove the following two results:

(a) Any point on the perpendicular bisector of a line segment is equidistant from the two end-points of the line segment and its converse.

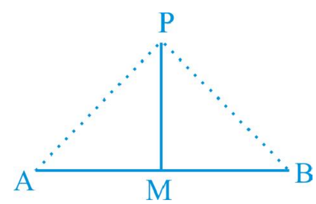
(b) Any point on the bisector of an angle is equidistant from the two arms of the angle and its converse.

The converse of part (a) is done for you in the dialogue form as shown below:

T : For the converse, we have to show that if a point P is equidistant from end-points A and B of the line segment AB, then P lies on the perpendicular bisector of AB. So, let us take a line segment AB with mid-point M and join it with point P, which is equidistant from A and B (see Figure). What we need to do now?

S_1 : We need to show that PM $\perp$ AB and MA = MB.

T : Are we given anything already?



S_2 : Yes, MA = MB (as M is the mid-point)

T : What else to be done?

S_3 : To show that PM $\perp$ AB.

T : What do you suggest?

S_4 : Join PA and PB.

T : What tempted to do this?

S_4 : Because we are given that PA = PB.

T : What have you obtained now?

S_5 : Two triangles PMA and PMB.

T : Can we say, they are congruent?

S_5 : We shall have to examine.

T : So, let us look at ΔPMA and ΔPMB. What do you observe?

S_6 : PA = PB (Given)

S_7 : MA = MB (Given)

S_8 : And PM = PM (Common)

S_9 : So, $\Delta PMA \cong \Delta PMB$ (by SSS).

T : What can we infer now?

S_1 : $\angle PMA = \angle PMB$ (CPCT)

S_2 : But $\angle PMA + \angle PMB = 180°$

T : It means $\angle PMA + \angle PMA = 180°$.

S_{10}: Yes, Madam. And so, $2\angle PMA = 180°$.

S_1 : Hence $\angle PMA = 90°$, i.e., PM $\perp$ AB.

T : It means PM is perpendicular to AB and also it bisects AB. Thus, P lies on the perpendicular bisector of the line-segment *AB*.

Q : Prove part (a), part(b) and its converse, using the dialogue approach.

The importance of these results lies in the fact that many times, they are themselves used as theorems in proving other results.

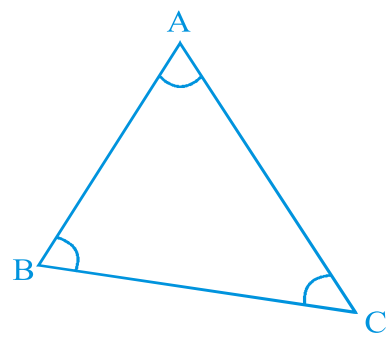
(iii) Inequalities in a Triangle

In the chapter on 'Triangles,' following three inequality relations are given:

(a) In a triangle, (a) angle opposite the longer side is greater.

(b) Side opposite to the greater angle is longer and

(c) Sum of any two sides is greater than the third side.



Let us consider the result (a).

T : What are we given?

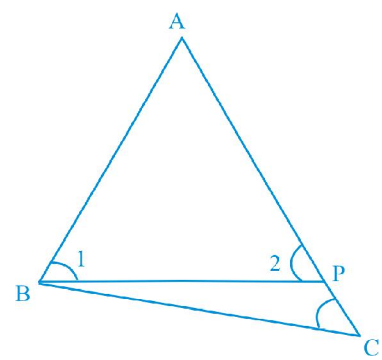
S_1 : A triangle ABC in which AC > AB. (See figure)

T : What is to be proved?

S_1 : Angle opposite AC ($\angle$B) is greater than angle opposite AB ($\angle$C). That is, $\angle B > \angle C$.

T : You are given AC > AB. Can you take a point P on side AB such that AP = AC?

S_2 : No. Madam



T : Can you take a point P on side AC such that AP = AB?

S_2 : Yes, Madam. It is possible.

T : Take such a point P on AC and join BP (See figure). What have you obtained?

S_3 : A triangle ABP in which AB = AP.

T : What can you say about $\angle 1$ and $\angle 2$?

S_3 : $\angle 1 = \angle 2$

T : Why?

S_4 : Angles opposite the equal sides are equal.

T : Is $\angle 2 = \angle C$?

S_4 : No, Madam.

T : Why?

S_4 : Because $\angle 2$ is an exterior angle of Δ PBC and an exterior angle is always greater than each of its interior opposite angles.

T : So, can you say that $\angle 1 > \angle C$?

S_4 : Yes, Madam, because $\angle 1 = \angle 2$.

T : Can you say that $\angle 1 + \angle PBC > \angle C$?

S_5 : Obviously, yes. In fact, $\angle 1 + \angle PBC$ will be more greater than $\angle C$.

T : What is ∠1 + ∠PBC?

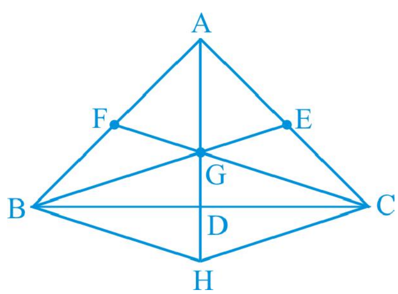
S_6 : It is ∠B.

T : So, is it not true that ∠B > ∠C ?

S : Perfectly true, Madam.

Review Questions

1. Prove the above results (b) and (c), through the dialogue approach.
2. Most of the results on special types of quadrilaterals such as parallelograms, rectangles, etc., are based on the congruency of the two triangles. Try to get these results proved in dialogue form.
3. Prove the mid-point theorem and its converse in the dialogue mode.
4. Through the dialogue form, prove that medians of a triangle meet at a point and this point (called the **centroid** of the triangle) divides each median in the ratio 2:1. [Take help of the adjoining figure, in which BE and CF are medians and AG = GH]



(iv) Similarity of Figures

Another important concept related to daily life in geometry is the concept of similarity of figures. It should gradually be explained out of the already learnt concept of congruency. It may be observed that two figures are said to be congruent, if they have the same shape and same size. However, two figures are said to be similar, if they have the same shape. Their sizes may or may not be the same. Thus, all congruent figures are similar but the converse is not true. Enlargement of different photographs, maps of a country or world in different sizes, etc. are good examples of similarity of figures. This impression should be removed from the minds of the students that:

(i) Similarity exists only in plane figures.

(ii) Similarity exists only in triangles.

In fact. we do have similarity in three dimensional figures also. For example, all cubes are similar, all regular tetrahedrons are similar, etc.

Further, we have similarity in other figures, also. For example, all squares are similar, all regular polygons of the same number of sides are similar, all circles are similar and so on.

Through activities, as suggested in the textbook of Class X Mathematics (NCERT), you must explain that two polygons are said to be similar if

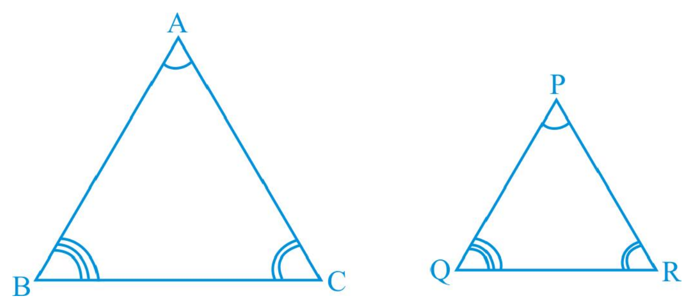
(i) their corresponding angles are equal

(ii) their corresponding sides are proportional.

However, in the case of triangles, it has been proved that one condition out of the above two conditions implies the other and hence only one condition is sufficient to establish the similarity of two triangles which are known as AAA similarity criterion and SSS similarity criterion. These results are proved on applying the Corollaries relating to *Basic Proportionality Theorem (BPT)* (also known as *Thales Theorem)* and its converse.

AAA similarity criterion may be proved in a dialogue form as shown below:

T : We are given two triangles ABC and PQR such that $\angle A = \angle P$, $\angle B = \angle Q$ and $\angle C = \angle R$ and we are to prove that these triangles are similar (see figure). For obtaining similarity of triangles, what other condition is required?



S_1 : Corresponding sides must be proportional.

T : What does it mean?

S_1 : Sir, it means $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP}$.

T : How to proceed? What can we say about AB and PQ?

S_2 : There are three possibilities:

(i) AB = PQ,

(ii) AB < PQ and

(iii) AB > PQ.

T : So, take first possibility, i.e, AB = PQ.

S_3 : If AB = PQ, then $\Delta ABC \cong \Delta PQR$ by ASA congruence criterion.

T : What does it mean?

S_3 : It means that $\Delta ABC \sim \Delta PQR$.

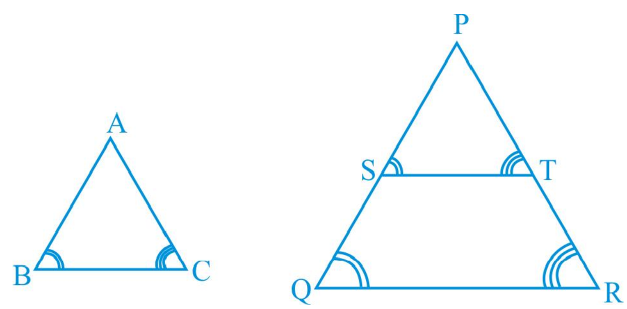
T : Why?

S_3 : All congruent figures are similar.

T : OK. Now, take the second possibility, i.e., AB < PQ. Can you take a point S on PQ such that AB = PS?

S_3 : Yes, Sir.

T : Through S, draw ST || QR (See figure).



S_3 : I have drawn ST || QR such that T is on PR.

T : What can you say about $\angle S$ and $\angle Q$?

S_4 : They are equal.

T : Why?

S_4 : Because, they are corresponding angles, as transversal QP intersects parallel lines ST and QR.

T : What about $\angle T$ and $\angle R$?

S_4 : They are also equal, because they are again corresponding angles.

T : But $\angle B = \angle Q$ is given. What can we say about $\angle B$ and $\angle S$?

S_5 : $\angle B = \angle S$

T : What about $\angle C$ and $\angle T$?

S_5 : Similarly, $\angle C = \angle T$.

T : Can you say that $\Delta ABC \cong \Delta PST$?

S_6 : Yes, Sir.

T : By which criterion of congruency?

S_6 : By ASA congruence criterion.

T : If $\Delta ABC \cong \Delta PST$, what can we say about their sides?

S_6 : AB = PS, AC = PT and BC = ST (CPCT)

T : You have drawn ST ∥ QR.

S_7 : Yes, Sir.

T : So, what can you say about $\frac{PS}{PQ}$ and $\frac{PT}{PR}$?

S_7 : $\frac{PS}{PQ} = \frac{PT}{PR}$

T : Why?

S_7 : By Corollary of BPT.

T : Can you now say that $\frac{AB}{PQ} = \frac{AC}{PR}$?

S_7 : Yes, Sir because AB = PS and AC = PT.

T : Similarly, by taking a suitable point on BC, you can have $\frac{AB}{PQ} = \frac{BC}{QR}$.

S_8 : So, we have obtained $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR}$.

T : What does it mean?

S_8 : It means that corresponding sides of the two triangles are proportional.

T : It means $\Delta ABC \sim \Delta PQR$.

S_9 : Yes, Sir, because both the conditions are satisfied.

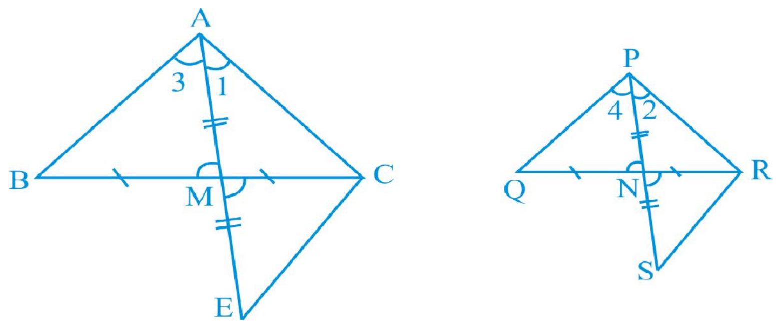
T : The case AB > PQ can be similarly treated. I hope you have enjoyed the learning.

S : Yes, Sir.

A Problem for Discussing Problem Solving Techniques (Also see Unit 6)

Problem: ABC and PQR are two triangles such that $\frac{AB}{PQ} = \frac{AC}{PR} = \frac{AM}{PN}$, where AM and PN are respectively the medians of the two triangles. Prove that $\Delta ABC \sim \Delta PQR$.

T : First draw a figure as per conditions of the problem as follows:



S_1 : We have drawn the figure.

T : For showing the similarity of triangles, we need to show that $\frac{AB}{PQ} = \frac{BC}{QR}$.

S_1 : How to link $\frac{AB}{PQ}$ with $\frac{BC}{QR}$?

T : It means that we should replace $\frac{AM}{PN}$ by $\frac{BC}{QR}$. That is, we should first try to relate $\frac{AM}{PN}$ and $\frac{BC}{QR}$.

S_2 : It is really not an easy task. We should have to think.

S_3 : I remember that while solving a problem relating to medians, we had produced median say AD to a point E such that AD = DE and join EC and obtained AB = CE.

T : Let us do the same in both these triangles ABC and PQR. So, produce AM to E such that AM = ME and produce PN to S such that PN = NS. Then join EC and SR (see the figure).

S_1 : $\Delta ABM \cong \Delta ECM$ (SAS) and

$\Delta PQN \cong \Delta SRN$ (SAS)

S_2 : So, AB = CE and PQ = RS (CPCT)

T : Is $\frac{AB}{PQ} = \frac{CE}{RS}$?

S : Yes

T : Why?

S_3 : Because AB = CE and PQ = RS.

S_4 : Similarly, $\frac{AM}{PN} = \frac{2AM}{2PN}$

i.e., $\frac{AM}{PN} = \frac{AE}{PS}$

T : It means that in triangles ACE and PRS, we have: $\frac{CE}{RS} = \frac{AC}{PR} = \frac{AE}{PS}$

S_5 : So, $\Delta ACE \sim \Delta PRS$

T : By which criterion ?

S_5 : By SSS similarity criterion.

T : Can you say that $\angle 1 = \angle 2$?

S_6 : Yes, Sir.

T : Why?

S_6 : Corresponding angles of similar triangles.

T : Can you also show that $\angle 3 = \angle 4$?

S_7 : Yes, Sir.

T : How?

S_7 : By joining BE and QS and establishing that $\Delta AMC \cong \Delta EMB$ and $\Delta PNR \cong \Delta SNQ$.

T : So, $\angle 1 + \angle 3 = \angle 2 + \angle 4$

S_7 : Yes, Sir.

T : It means that $\angle BAC = \angle QPR$.

S_8 : Yes, Sir.

T : So, in ΔABC and ΔPQR, $\frac{AB}{PQ} = \frac{AC}{PR}$ and $\angle BAC = \angle QPR$

S_8 : Yes, Sir.

T : Can you now say that $\Delta ABC \sim \Delta PQR$?

S_9 : Yes, Sir.

T : By which criterion?

S_9 : By SAS similarity criterion.

T : Thus, we have proved the result.

S : Yes, Sir. Our technique has shown result.

Review Questions

1. Prove the other two criteria of similarity namely SSS and SAS similarity criteria, adopting the dialogue approach.
2. Explain, the importance of writing the similarity of two triangles in the symbolic form with correct correspondence of vertices.
3. Can there be two triangles in which their five pairs of parts are equal but they are not congruent? If yes, draw such a pair. If no, draw one such pair of triangles.

(v) Pythagoras Theorem

Here, it may be emphasised that this theorem has been proved in a number of ways. One proof using the property. "If a triangle and a parallelogram lie on the same base and between the same parallels, then *ar*(parallelogram) = 2*ar*(triangle)" has been discussed in Class IX Mathematics textbook of NCERT and a proof using the idea of similarity' has been discussed in Class X Mathematics Textbook of NCERT. In addition, a number of activities have also been provided in Class VII Mathematics Textbook, Laboratory Mannuals for different stages of schooling published by NCERT. All the above may be examined by the students carefully, which will provide them a better understanding of the theorem. Teacher may also discuss the statement provided by the ancient Indian Mathematician Baudhayan (800 BC) and explain its relevance with the Pythagoras Theorem. After this, the application of similarity along with Pythagoras Theorem may be discussed, more specific in the study of Trigonometry. It may also be explained that in the statement of the theorem, 'Square' can be replaced by any regular polygon or semicircle everywhere. For example, in a right triangle, sum of the 'equilateral triangles' on the two sides is equal to the equilateral triangle formed on the hypotenuse, etc.

Review Questions

1. Which of the following statements are true and which are false?
   (i) In a right triangle, sum of the rectangles on the two sides is equal to the rectangle formed on the hypotenuse. (False)
   (ii) In a right triangle, sum of the semicircles on two sides is equal to the semicircle on the hypotenuse. (True)
2. In Q.No. 5 of Exercise 6.6 of Class X Mathemaitcs Textbook of NCERT, it has been stated that for a triangle ABC with AD as a median, $AB^2 + AC^2 = 2AD^2 + \frac{1}{2}BC^2$.
   (An extension of Pythagoras Theorem).
   Use this result and provide an alternative proof for the previously discussed problem given below, using dialogue mode:

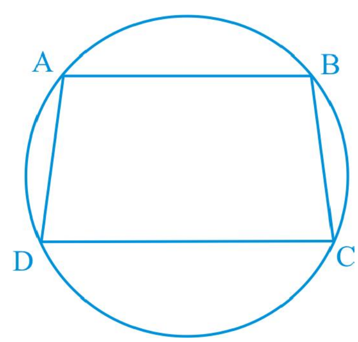
"If AM and PN are medians of two triangles ABC and PQR such that $\frac{AB}{PQ} = \frac{AC}{PR} = \frac{AM}{PN}$, then prove that $\Delta ABC \sim \Delta PQR$."

(vi) Circle

Though the students are familiar with the concepts of a circle and its parts such as radius, centre, diameter, circumference, chord, arc, sector, etc. from the very beginning, but its formal geometrical study starts from Class IX itself and completed in Class X, with tangents to a circle. Each and every property discussed in Mathematics Textbooks of Classes IX and X should be explained through suitable examples before applying them in solving problems. It can be seen that at most of the places, the knowledge of congruence of triangles and the Pythagoras theorem is must. Further, all the properties are very closely related to each other and hence problems related to them can be either directly solved using a single theorem or sometimes by the combination of two or more theorems. Here, again the problem solving techniques (also See Unit 6) may be explained through dialogue mode as shown below:

Problem 1: Prove that a cyclic trapezium is always isosceles.

T : Let us consider a cyclic trapezium ABCD in which AB || DC (See figure). We wish to show that it is isosceles, i.e., AD = BC. How to proceed?



S_1 : $\angle A + \angle C = 180°$

T : Why?

S_1 : $\angle A$ and $\angle C$ are opposite angles of a cyclic quadrilateral ABCD.

T : OK. What can you infer from AB || DC ?

S_2 : $\angle A + \angle D = 180°$

T : Why?

S_2 : Interior angles on the same side of the transversal

T : So, can you say that $\angle C = \angle D$?

S_3 : Yes, Sir.

T : How ?

S_3 : From $\angle A + \angle C = \angle A + \angle D$ (As both = 180°)

T : But we want AD = BC.

S_4 : Let us obtain two triangles involving AD and BC.

T : How ?

S_4 : Draw $AM \perp DC$ and $BN \perp DC$ (See Fig.)

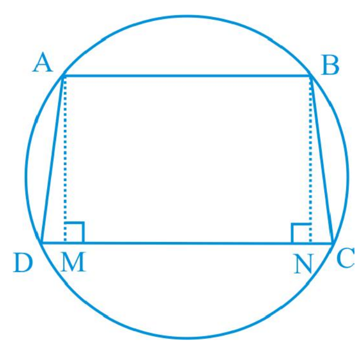
T : Which two triangles have been obtained ?

S_4 : ΔADM and ΔBCN.

T : Very good. Now, let us examine these for congruence.

S_5 : AM = BN

T : Why ?



S_5 : Distance between two parallel lines is the same everywhere.

T : What else ?

S_5 : $\angle AMD = \angle BNC$ (each = 90°)

and $\angle D = \angle C$ (Already shown)

T : Can you now say that $\Delta ADM \cong \Delta BCN$?

S_6 : Yes, Sir.

T : By which criterion ?

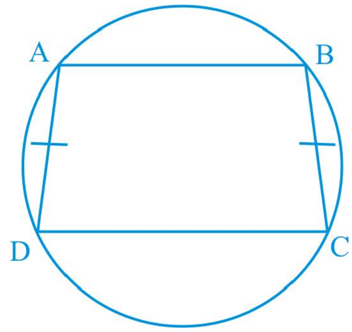
S_6 : By AAS criterion.

S_7 : So, AD = BC

T : Why ?

S_7 : By CPCT.

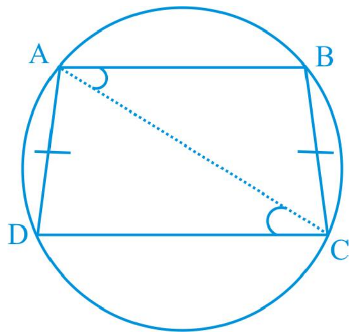
Problem 2: ABCD is a cyclic quadrilateral in which AD = BC. Prove that ABCD is a trapezium.



T : We are given that ABCD is a cyclic quadrilateral in which AD = BC (See figure). We need to show that AB || DC. How to proceed?

S_1 : For showing AB || DC, we should try to have a pair of alternate angles.

S_2 : Join AC. We have obtained ∠ACD and ∠BAC.

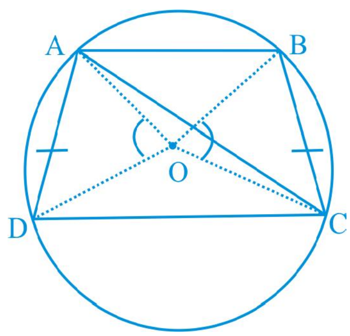


T : Are they equal?

S_2 : We don't know. Let us examine.

T : Take the centre of the circle 'O'

S_3 : Let us join OA, OB, OC and OD.



T : Can you say that ∠AOD = ∠BOC ?

S_3 : Yes.

T : Why?

S_3 : Because AD = BC and equal chords of a circle subtend equal angles at the centre.

T : Is ∠ACD related to ∠AOD ?

S_4 : Yes, Sir, $\angle ACD = \frac{1}{2}\angle AOD$

T : Why ?

S_4 : Angle at the centre is double the angle at any other point on the circle.

T : Similarly, $\angle BAC = \frac{1}{2}\angle BOC$. Is it not ?

S_4 : Yes, Sir. By same argument.

S_5 : Therefore, ∠ACD = ∠BAC

T : How ?

S_5 : Because ∠AOD = ∠BOC, already shown.

T : Then, what can be said of AB and DC ?

S_5 : AB ‖ DC

T : Why ?

S_5 : Because ∠ACD and ∠BAC are alternate angles.

At this stage, it may be highlighted that **in case of two equal chords of a circle, we can directly say that angles subtended by them on remaining parts of the circle are equal, because of the two properties namely**

(i) equal chords subtend equal angles at the centre and

(ii) angle subtended by an arc at the centre is double the angle subtended by the arc at any point on the remaining part of the circle.

Problem 3: P is any point on the minor arc BC of the circumcircle of an equilateral ΔABC. Prove that AP = BP + CP.

T : What is an equilateral triangle ?

S_1 : A triangle whose all the three sides are equal. Thus, in ΔABC, AB = BC = CA and each angle is 60°.

T : What is circumcircle of the triangle ?

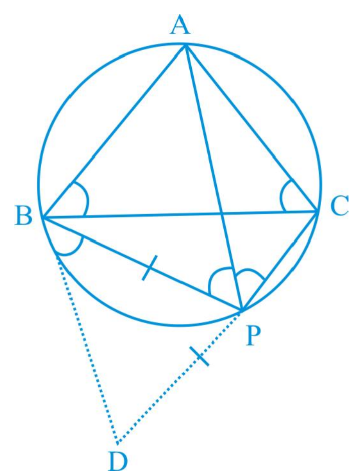
S_2 : A circle which passes through the three vertices of the triangle. Thus, circumcircle of ΔABC will pass through A, B and C.

T : OK. What is this minor arc BC ?

S_3 : In the circle, there are, in general two arcs BC, out of which one is smaller and one is greater. The smaller one is the minor arc.

T : Very good. With this knowledge, can you draw the figure ?

S_4 : Yes, Sir. It is drawn here (See figure).



T : Can you now prove that AP = BP + CP ?

S_4 : No.There is a difficulty.

T : What is the difficulty ?

S_4 : On LHS, **there is only one line segment AP and on the RHS, there is a sum of two line segments BP and CP.**

T : How to overcome this difficulty ?

S_5 : We should try to obtain a single line segment equal to BP + CP.

T : How can it be done ?

S_6 : Produce CP to a point D such that PD = BP (See figure).

S_7 : Thus, we have obtained CD = CP + PD = BP + CP.

S_6 : It means that we have obtained a single line segment CD equal to BP + CP.

T : Now, what is to be done ?

S_6 : We have to show that AP = CD.

T : How to do this ?

S_7 : We should look for two triangles involving these two sides AP and CD.

T : We have a ΔABP involving AP as one side. How to get a triangle with CD as one side?

S_8 : Join BD (See figure). We have a ΔCBD, involving CD as one side.

T : To establish AP = CD, what should we do.

S_8 : We should try to show that triangles ABP and CDB are congruent.

T : Let us see, how can we do it.

S_8 : Let us start with our construction, namely BP = PD.

S_9 : Since BP = PD, so $\angle PBD = \angle PDB$.

S_{10}: Also, $\angle APB = \angle ACB = 60°$ (Angles in the same segment) and $\angle APB = \angle ABC = 60°$ (Angles in the same segment)

T : What can we say about $\angle BPD$?

S_{10}: It is equal to $180° – 60° – 60° = 60°$

T : So, what is $\angle PBD + \angle PDB$?

S_{10}: It is $180° – \angle BPD$, i.e., $180° – 60° = 120°$

T : But $\angle PBD = \angle PDB$ have been shown by you. So, what else can you say now ?

S_9 : We can say that $\angle PBD + \angle PDB = 120°$, i.e., $\angle PBD = 60°$ and also $\angle PDB = 60°$. Hence , we have BD = BP = PD.

T : Then, what is $\angle PBD + \angle PBC$?

S_9 : It is $\angle CBD = 60° + \angle PBC$.

T : What about $\angle ABP$?

S_{10}: It is $\angle ABC + \angle PBC$, which is equal to $60° + \angle PBC$.

T : What can you say about $\angle ABP$ and $\angle CBD$?

S_{10}: They are equal, i.e., $\angle ABP = \angle CBD$.

T : Can you now show that $\Delta ABP \cong \Delta CBD$?

S_{10}: Yes, sir.

T : How ?

S_{10}: In ΔABP and ΔCBD, AB = CB, (Sides of equilateral triangle ABC)

$\angle ABP = \angle CBD$ (Shown above)

and BP = BD (by construction)

Hence, $\Delta ABP \cong \Delta CBD$ (By SAS)

T : What can you conclude from this?

S_{10}: AP = CD (CPCT)

and hence AP = BP + CP.

Review Question

Prove that two tangents from an external point to a circle are equal,using the dialogue approach.

Problem Posing Techniques

Like 'problem solving', 'problem posing' is also an 'art'. In the classroom, problems must be posed keeping in view the maturity level of the students. A teacher is the best judge regarding the level of his/her students. Following points may be kept in mind for problem posing:

(i) Language of the problem (question) should be within the comprehension level of the students. If necessary, it may be diluted or initially the problem may be posed along with a figure.

(ii) In the beginning, only 'one step' problems' (or riders or questions) may be given and gradually proceed to 'two steps' problems, 'three steps' problems and then to more difficult problems.

(iii) If necessary, a long or difficult problem (or question) may be presented in a different form by just spliting it into few smaller steps. As suggested samples, some problems are given below:

Problem 1: If arms of two angles are parallel separately to each other, then prove that the two angles are either equal or supplementry.

Suggested split up: It may be splitted and posed as follows:

(i) In Fig. (i), BA || ED and BC || EF. Prove that $\angle ABC = \angle DEF$.

(ii) In Fig. (ii), BA || ED and BC || EF. Prove that $\angle ABC + \angle DEF = 180°$

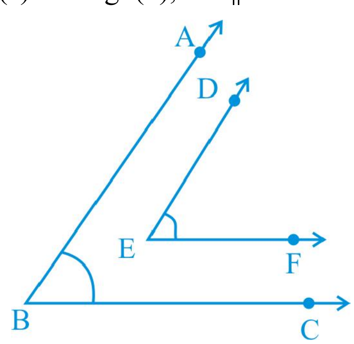


Fig. (i)

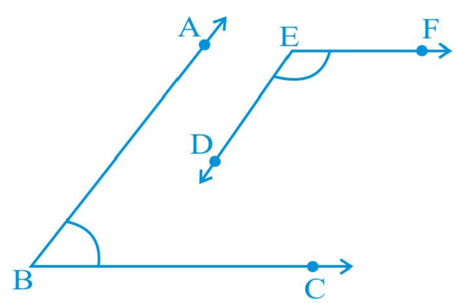


Fig.(ii)

Problem 2: Show that the perimeter of a quadrilateral is less than twice the sum of its diagonals.

Suggested split up: ABCD is a quadrilateral whose diagonals intersect each other at the point O (See figure). Show that

(i) OA + OB > AB

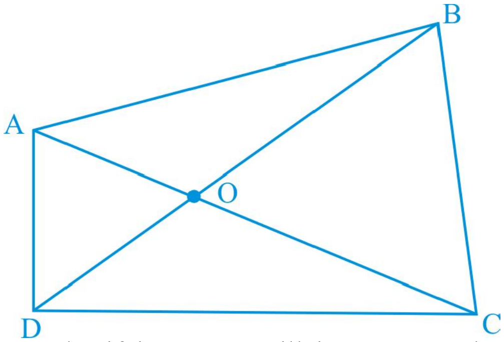
(ii) OB + OC > BC

(iii) OC + OD > CD

(iv) OD + OA > AD

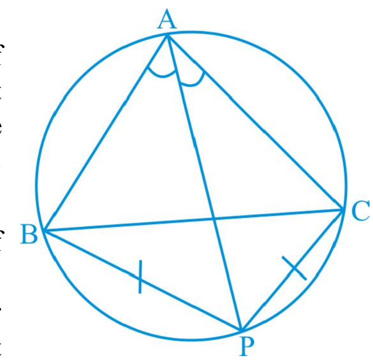
(v) (OA + OB + OB + OC + OC + OD + OD + OA) > AB + BC + CD + AD

(vi) 2(AC + BD) > AB + BC + CD + AD



Problem 3: Prove that the bisector of an angle of a triangle and the perpendicular bisector of the side opposite that angle, if intersect, will intersect at the cirmumcircle of the triangle.

Suggested split up (1): Bisector of $\angle A$ of ΔABC intersects the circumcircle of ΔABC at the point P. Prove that PB = PC, i.e., P lies on the perpendicular bisector of side BC (See figure).



Suggested split up (2): P is a point on the circumcircle of a ΔABC such that PB=PC, i.e., P lies on the perpendicular besector of side BC. Prove that $\angle PAB = \angle PAC$, i.e., AP is the bisector of $\angle BAC$ (See figure).

Review Question

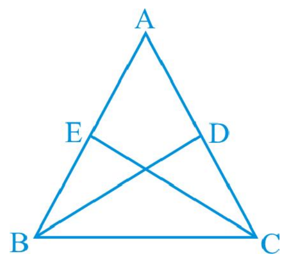
1. Show that the perimeter of a quadrilateral is greater than the sum of its diagonals.

For the above, question, suggest a suitable split up for problem posing.

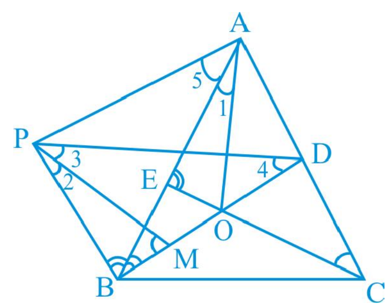
Enrichment Materials

Here, some material is being provided which is normally not available in the school mathematics textbook and curriculum. It is for the interested students only and with the intention that one should not think it to include these materials as the part of school mathematics curriculum. It is only suggestive and not exhaustive.

(i) In the adjoining figure, BD and CE are the bisectors of ∠B and ∠C respectively of a ΔABC such that BD = CE. Prove that ABC is an isosceles triangle.



Solution: Construct ΔPBD ≅ ΔAEC. Also, draw bisector of ∠BPD to intersect BD at M and join A with the point of intersection O of BD and CE. Join AP (See figure).



As ΔPBD ≅ ΔAEC, therefore ∠BPD = ∠BAD (CPCT) and ∠PBD = ∠AEC (CPCT). Therefore, APBD is a cyclic quadrilateral. (Why?)

Also, AO is the bisector of ∠BAC (The three angle bisectors of a triangle meet at a point)

So, $\angle 1 = \frac{1}{2} \angle A$

Also, $\angle 2 = \frac{1}{2} \angle BPD$

So, $\angle 1 = \angle 2$.

Now, in ΔPBM and ΔAEO, we have:

PB = AE (Because ΔPBD ≅ ΔAEC, CPCT)

$\angle 1 = \angle 2$

and $\angle PBM = \angle AEO$ $(\angle PBD = \angle AEC)$

So, $\Delta PBM \cong \Delta AEO$ (ASA criterion)

Hence, $PM = AO$ (CPCT) (1)

Now, $\angle 5 = \angle 4$ (PADB a cyclic quadrilateral)

and $\angle 1 = \angle 3$ (Each is equal to $\frac{1}{2}\angle A$)

So, $\angle 5 + \angle 1 = \angle 4 + \angle 3$

But from ΔPMD, $\angle 4 + \angle 3 + \angle PMD = 180°$

So, $\angle 5 + \angle 1 + \angle PMD = 180°$

or $\angle PAO + \angle PMD = 180°$

That is, in quadrilateral PAOM, a pair of opposite angles is supplementory.

So, quadrilateral PAOM is a cyclic quadrilateral. Of this quadrilateral, opposite sides PM and AO are equal. [Proved in (1)]

So, PAOM is a trapezium in which PA || MO (See Problem 2 of 'problem solving techniques' discussed in circles).

Therefore, PA || BD

So, $\angle 5 = \angle ABD$ (Alternate angles)

i.e., $\angle 5 = \frac{1}{2}\angle B$ (Because BD is bisector of $\angle B$) (2)

Also, $\angle 5 = \angle 4$ (PADB is cyclic)

and $\angle 4 = \angle ACE$ ($\Delta PDB \cong \Delta ACE$, CPCT)

But $\angle ACE = \frac{1}{2}\angle C$ (CE is bisector of $\angle C$)

So, $\angle 4 = \frac{1}{2}\angle C$ (3)

Hence, from (2) and (3),

$$\frac{1}{2}\angle B = \frac{1}{2}\angle C$$

or $\angle B = \angle C$

Hence, $AB = AC$ (Sides opposite to equal angles)

That is, ΔABC is isosceles.

(ii) Proof by Exhaustion for Pythagoras Theorem

Given: ABC is a right triangle, right angled at B (See figure).

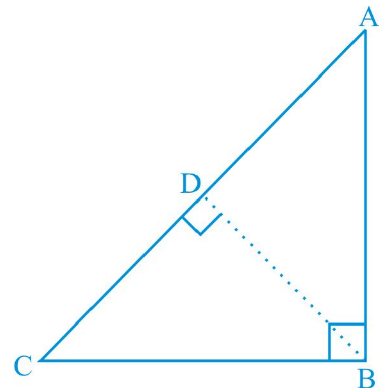
To prove: $AC^2 = AB^2 + BC^2$

Construction: Draw $BD \perp AC$

Proof: Let $AC^2 \neq AB^2 + BC^2$

So, either (i) $AC^2 > AB^2 + BC^2$, or

(ii) $AC^2 < AB^2 + BC^2$



(i) By assuming $AC^2 > AB^2 + BC^2$, we mean that in every right triangle, square on hypotenuse is greater than the sum of the squares on the other two sides.

So, from ΔABC,

$$AC^2 > AB^2 + BC^2 \quad (1)$$

From ΔADB,

$$AB^2 > AD^2 + BD^2 \quad (2)$$

and from ΔBDC,

$$BC^2 > BD^2 + DC^2 \quad (3)$$

Adding (2) and (3),

$$AB^2 + BC^2 > AD^2 + DC^2 + 2BD^2 \quad (4)$$

Now, we have

$\Delta ADB \sim \Delta BDC$ (Perpendicular drawn from the vertex of a right triangle to the hypotenuse divides the triangle into two similar triangles).

So, $\dfrac{AD}{BD} = \dfrac{BD}{CD}$

or $AD \times DC = BD^2$ (5)

So, from (4) and (5),

$AB^2 + BC^2 > AD^2 + DC^2 + 2\, AD \times DC$

or $AB^2 + BC^2 > (AD + DC)^2$

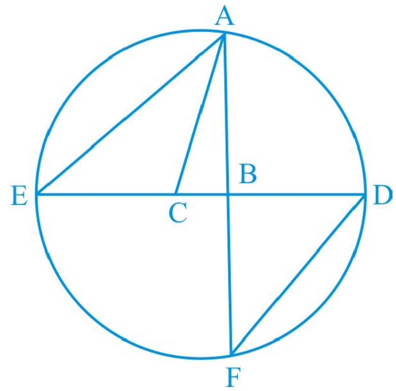
or $AB^2 + BC^2 > AC^2$, which contradicts the assumption that $AC^2 > AB^2 + BC^2$

Similarly, by taking the assumption that $AC^2 < AB^2 + BC^2$, we shall arrive at $AC^2 > AB^2 + BC^2$, which contradicts the assumption.

Hence, there is only one possibility, i.e., $AC^2 = AB^2 + BC^2$

This proof is sometimes also referred to as an *Inequality Proof of Pythagoras Theorem*.

(iii) Proving Converse of Pythagoras Theorem, without using Pythagoras Theorem



Let ABC be a triangle in which $AC^2 = AB^2 + BC^2$.

We are to prove that $\angle ABC = 90°$.

Construction: With C as centre and CA as radius, draw a circle. Let CB produced meet the circle at points D and E. Also, let AB produced meet the circle at F.

Join AE and DF (See figure).

Proof: We are given: $AC^2 = AB^2 + BC^2$

So, $AC^2 - BC^2 = AB^2$

or $(AC + BC)(AC - BC) = AB^2$

or $(CE + BC)(CD - BC) = AB^2$ (because AC = CE = CD)

or $BE \times BD = AB^2$ (1)

Now, we have:

$\Delta ABE \sim \Delta DBF$ ($\angle E = \angle F$, $\angle ABE = \angle DBF$, AA similarity)

So, $\dfrac{AB}{BD} = \dfrac{BE}{BF}$

or $BD \times BE = AB \times BF$ (2)

Form (1) and (2), we have:

$AB^2 = AB \times BF$

or $AB \times AB = AB \times BF$

or $AB = BF$

So, B is the mid-point of chord AF of the circle.

Therefore, $CB \perp AF$ (Line segment joining the mid-point of a chord to the centre is perpendicular to the chord)

So, $\angle ABC = 90°$

(iv) Euler's Line

Prove that, in general, the orthocentre, centroid and circumcentre of a triangle lie on a line and the centroid divides the line segment joining orthocentre and circumcentre in the ratio 2:1 (This line is called the **Euler's Line**).

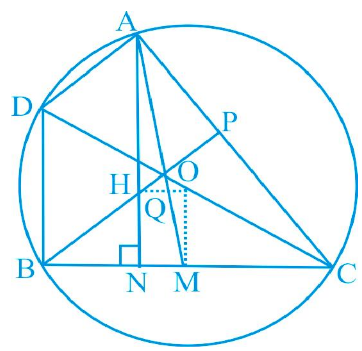
Solution: For solving this problem, first we shall have to prove the following result:

The distance of the orhocentre H of a triangle ABC from the vertex A is double the distance of the cicumcentre O of the triangle from the opposite side BC of vertex A (See the figure). H is the ortho-centre and O is the circumcentre of ΔABC. Let COD be the diameter of the circumcircle of the ΔABC.

Construction: Join BD, produce AH to meet BC at N and BH to meet AC at P. Also, join AD.

Proof: As H is orthocentre, so AN ⊥ BC and BP ⊥ AC.

Now, in ΔCDB, O is the mid-point of CD and M is the mid-point of BC.



So,	BD ∥ OM and BD = 2 OM	(Mid-point theorem)	(1)
Also,	AN ∥ OM	(Both are ⊥ to BC)	
So,	BD ∥ AN, i.e., BD ∥ AH		(2)
Again,	∠DAC = 90°	(Angle in a semicircle)	
So,	DA ⊥ AC		
Also,	BP ⊥ AC		
So,	DA ∥ BP, i.e. DA ∥ BH		(3)
Therefore,	BDAH is a parallelogram	[From (2) and (3)]	
So,	BD = AH	(opposite sides of a parallelogram)	(4)
But	BD = 2OM	[From (1)]	
So,	AH = 2OM		(5)

Now, let us join AM to intersect HO at Q. Consider ΔAHQ and ΔMOQ.

AH ∥ OM

So, ∠HAQ = ∠OMQ (Alternate angles)

Also, $\angle AQH = \angle MQO$ (Vertically opposite angles)

So, $\Delta AHQ \sim \Delta MOQ$ (AA similarity)

Therefore, $\dfrac{AQ}{MQ} = \dfrac{HQ}{OQ} = \dfrac{AH}{OM}$

But from (5), $\dfrac{AH}{OM} = \dfrac{2}{1}$

Therefore, $\dfrac{AQ}{MQ} = \dfrac{HQ}{OQ} = \dfrac{2}{1}$

That is, $\dfrac{AQ}{MQ} = \dfrac{2}{1}$ (6)

and $\dfrac{HQ}{OQ} = \dfrac{2}{1}$ (7)

Now, AM is a median of the triangle.

So, from (6), we have that Q is a point which divides median AM in the ratio 2:1.

Therefore, Q is the centroid of the triangle.

Thus, orthocentre, centroid and circumcentre (i.e., H, Q and O)lie on a line (which is known as Euler's line).

From (6), $\dfrac{HQ}{OQ} = \dfrac{2}{1}$, i.e., Q divides HO in the ratio 2:1. In other words, centroid divides the line segment joining the othocentre and the circumcentre in the ratio 2:1.

(v) Proving Ptolemy Theorem

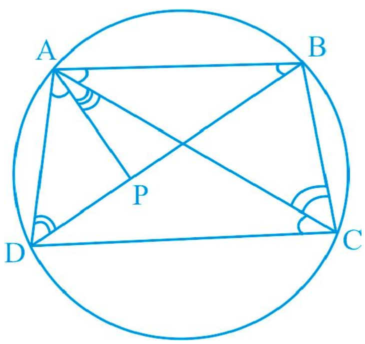
It states that product of the diagonals of a cyclic quadrilatral is equal to the sum of the products of its opposite sides.

Given: ABCD is a cyclic quadrilateral.

To prove: $AC \times BD = AB \times DC + AD \times BC$

Construction: Take a point P on BD such that $\angle DAP = \angle BAC$.

Proof: In ΔADP and ΔACB,



$\angle DAP = \angle BAC$	(Construction)
and $\angle ADP = \angle ACB$	(Angles in the same segment)
So, $\Delta ADP \sim \Delta ACB$	(AA similarity)

Hence, $\dfrac{AD}{AC} = \dfrac{DP}{BC}$

i.e., $BC \times AD = AC \times DP$ (1)

Now, $\angle DAP + \angle PAC = \angle BAC + \angle PAC$ (Why?)

So, $\angle DAC = \angle BAP$

Now, in ΔBAP and ΔCAD,

$\angle BAP = \angle DAC$	(Shown above)
and $\angle ABP = \angle ACD$	(Angles in the same segment)
So, $\Delta BAP \sim \Delta CAD$	(AA similarity)

Hence, $\dfrac{AB}{AC} = \dfrac{BP}{DC}$ or $AB \times CD = BP \times AC$.....(2)

Adding (1) and (2), we get $AD \times BC + AB \times CD = AC(BP + DP) = AC \times BD$

Try to explore some more such materials and discuss with your friends.

3.5 Misconceptions

- Many times students try to define the undefined terms-point, line and plane.
- Some students are not able to write the congruence of two triangles or similarity of two triangles in symbolic form with the correct correspondence and thus obtain wrong corresponding parts.

- Some students are not able to apply the correct congruency criterion in deciding the congruency of two triangles and correct similarity criterion in the case of deciding the similarity of two triangles.
- Some students state that if two parallel lines are intersected by a transversal, then the two interior angles on the same side of the transeversal are equal.
- Some students do not understand the proper use of two scales on the protractor. Some of them think that one scale is for drawing or measuring acute angles and the other is for drawing or measuring obtuse angles.
- Many students do not understand the meaning of ruler and compass construction. They have no idea that in these constructions, no marking is allowed on the ruler. Due to this misunderstanding, they claim to perform some constructions like trisecting an angle in their own way.
- **Fallacies:**

Fallacies are obtained due to wrong assumptions:

Example 1: Every triangle is isosceles.

Let ABC be any triangle.

Suppose the bisector of $\angle$A and the perpendicular bisector of side BC meet at O.

Draw OM $\perp$ AB and ON $\perp$ AC.

Join OB and OC.

O is a point on angle bisector AO.

So, OM = ON

Now, in Δ AMO and Δ ANO, we have:

OM = ON (Proved above)

OA = OA (Common)

So, ΔAMO $\cong$ ΔANO (RHS criterion)

Therefore, AM = AN (CPCT) (1)

Now, in ΔOBM and ΔOCN, we have:

OM = ON (Proved above)

OB = OC (O is a point on perpendicular bisector of BC)

So, ΔOBM $\cong$ ΔOCN (RHS criterion)

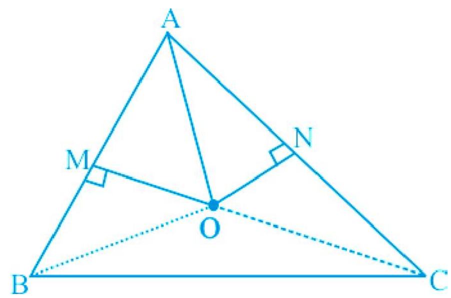
Therefore BM = CN (2)

Adding (1) and (2), we get

AM + BM = AN + CN

i.e., AB = AC

So, ΔABC is isosceles.



Thus, we have started with any triangle and arrive at the result that it is an isosceles triangle, which indicates the fallacy that every triangle is isosceles.

Think about this fallacy. Recall that perpendicular bisector of a side of a triangle and the bisector of the angle opposite this side intersect each other at the circumcircle of the triangle. Here, we have taken O in the interior of the triangle. Clearly, it should be on the circumcircle of ΔABC.

Example 2: An obtuse angle is equal to a right angle.

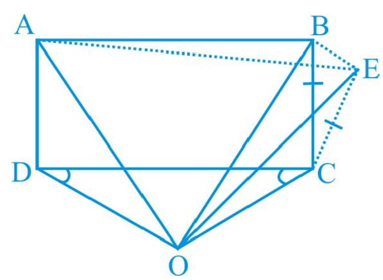
Let ABCD be a rectangle and let E be a point such that CB = CE. Join AE.

Let O be the point of intersection of the perpendicular bisectors of AB and AE.

Join OA and OE.

In ΔOAD and ΔOCE, we have:

OA = OE (O lies on the perpendicular bisector of AE)



OD = OC (O lies on the perpendicular bisector of AB and hence of CD)

and AD = CE (Because AD = BC and BC = CE)

Thus, ΔOAD ≅ ΔOCE (SSS criterion)

So, ∠ODA = ∠OCE (CPCT) (1)

Now, since OD = OC, therefore

∠ODC = ∠OCD (Angles opposite the equal sides) (2)

So, from (1) and (2),

∠ODA – ∠ODC = ∠OCE – ∠OCD

or ∠ADC = ∠ECD

i.e., **right angle = obtuse angle**

Think! What is wrong?

Note that O is equidistant from points A, B and E.

So, O must be the circumcentre of ΔABE. Only then, we shall get the correct figure. That is, O cannot be taken at the place as shown in the figure.

- **Wrong Proof Using Invalid Reasoning**

Sometimes, students apply wrong (invalid) reasoning in proving certain results. Some examples are given below:

(i) If in ΔABC, AB = AC and AD is the bisector of $\angle$A, prove that $\Delta ABD \cong \Delta ACD$.

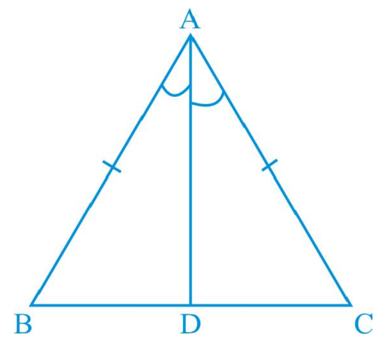
Proposed Proof: In ΔABD and ΔACD,

AB = AC (Given)

$\angle$B = $\angle$C (Since AB = AC)

$\angle$BAD = $\angle$CAD (As AD is bisector of $\angle$A)

So, $\Delta ABD \cong \Delta ACD$ (ASA congruence criterion)



This reasoning is not a valid reasoning, because if AB = AC, then $\angle$B = $\angle$C is proved by first proving that $\Delta ABD \cong \Delta ACD$. Therefore, for proving $\Delta ABD \cong \Delta ACD$ using $\angle$B = $\angle$C is not correct.

(ii) Prove that a diagonal of a parallelogram divides it into two congruent triangles.

Proposed Proof: ABCD is a parallelogram and AC is its diagonal.

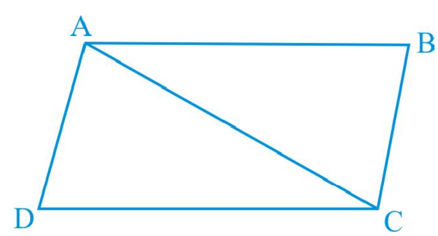
Now, In ΔABC and ΔCDA,

AB = CD (opposite sides of a parallelogram)

BC = DA(opposite sides of a parallelogram)

AC = CA(Common)

So, $\Delta ABC \cong \Delta CDA$ (SSS criterion)



This is also not a valid reasoning, because for proving "opposite sides of a parallelogram are equal", first it is proved that $\Delta ABC \cong \Delta CDA$ and then it is concluded that AB = CD and BC =DA as CPCT. Therefore , AB = CD and BC = DA.

(iii) **In a triangle, if a line is drawn parallel to one side to intersect the other two sides, then the other two sides are divided in the same ratio (BPT).**

Proposed Proof: Let ABC be a triangle in which a line $l \parallel$ BC intersects AB and AC at D and E respectively. We are to prove that $\frac{AD}{BD} = \frac{AE}{EC}$.

DE || BC.

So, $\angle 1 = \angle 2$ (Corresponding angles)

and $\angle 3 = \angle 4$ (Corresponding angles)

So, $\Delta ABC \sim \Delta ADE$ (AAA similarity criterion)

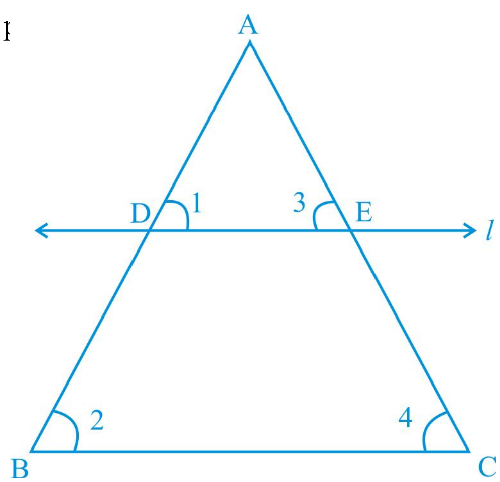
Therefore, $\dfrac{AB}{AD} = \dfrac{AC}{AE}$ (Sides are proportional)

or $\dfrac{AB}{AD} - 1 = \dfrac{AC}{AE} - 1$

or $\dfrac{AB - AD}{AD} = \dfrac{AC - AE}{AE}$

or $\dfrac{BD}{AD} = \dfrac{CE}{AE}$

i.e., $\dfrac{AD}{BD} = \dfrac{AE}{CE}$, Hence,



This is also not a valid reasoning, because, in general, for similarity of polygons, the following two conditions are necessary:

(i) Corresponding angles are equal.

(ii) Corresponding sides are proportional.

In case of similarity of two triangles, one condition implies the other, which have been proved using the Corollary of BPT or its converse. Hence, it is not correct to use the AAA similarity criterion for proving the above result which is nothing but the Basic Proportionality Theorem (BPT).

The reasoning shown in the above three examples is termed as **circular reasoning** which is quite common in geometry. Teachers may carefully observe this type of reasoning in their teaching-learning and may share their experiences with their colleagues and friends. This will bring a reasonable improvement in the study of mathematics in general and geometry in particular.

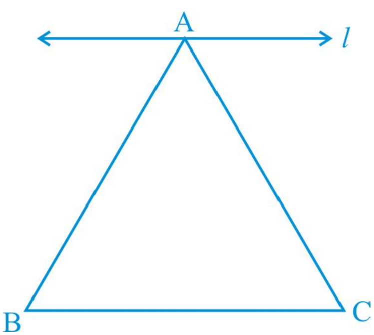
Circular Definitions: Like circular reasoning, one can find circular definitions in geometry such as given below:

(i) Right Angle: If two rays forming an angle are perpendicular, it is called a right angle

(ii) Perpendicular: Two rays or lines are called perpendicular to each other, if they form a right angle.

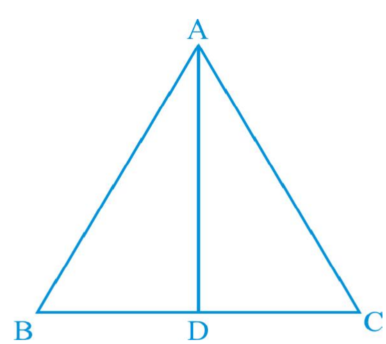
3.6 Assessment Techniques

Mathematics, in general and geometry in particular is a sequenced subject. For example, without the sound knowledge of basic concepts say lines and angles, students cannot proceed to triangles and congruence of triangles. Similarly, without proper understanding of congruency of triangles, one cannot proceed to similarity of triangles and circles. Therefore, the assessment of students must be continuous and comprehensive. So, it is necessary that after teaching of one or two concepts, students must be assessed before going to new concepts. As in geometry, teaching-learning shall be mainly of theorems, riders and constructions, these should be assessed during the teaching-learning process itself. Teacher should try to encourage students to write even the minute point involved in the solution of the problems solved by them. Special care must be taken while writing proofs of various results. Problems or questions must by properly selected for assessment and after getting the feedback on the basis of the assessment, suitable remedial instructions must be provided to the students. Some simple problems are suggested below:



3.7 Exercise

1. For proving that sum of the three angles of a triangle is 180°, we draw a line say l through A and parallel to side BC of ΔABC (Fig.). On what basis, you are allowed to draw this line l ?
2. In a triangle ABC with AB = AC, to prove $\angle$B = $\angle$C, we draw bisector of $\angle$A to meet BC at D. On what basis, you assume that bisector will intersect BC at one and only point D ?
3. A line can be denoted by taking any two points lying on it. Can you explain the reason for it?



UNIT
4

TEACHING OF MENSURATION

4.1 Introduction

Mensuration is a latin word which means measurement. In mensuration, we deal with the measurements of perimeters, areas and volumes of various shapes. Mensuration was originated from the need of measuring the land for cultivation. It was studied in various forms in ancient civilisations in Egypt, Babylonia, China, India, Greece, etc. as a special part of geometry. Whenever the construction of canals, pyramids, palaces, monuments, etc., was done by rulers, it was needed to calculate perimeters, areas and volumes of different geometrical shapes.

Now a days, mensuration is so much associated with every part of life, that one cannot imagine to live without its knowledge. The **lekhpals** in the villages use it to calculate areas of different fields and issue passbooks to the farmers mentioning the areas of their fields. This helps them to calculate the amount of seeds to sow, amount of manure to be used, etc. Engineers and architects calculate the cost of construction of a building knowing its dimensions. The knowledge of mensuration is also essential, in manufacturing many industrial items such as clothes, utensils, almirahs, soft drinks, petroleum products.

4.2 General Strategies

1. Explaining the idea of areas of plane figures by using concrete objects.
2. Explaining the idea of volumes of solids by concrete objects as well as rough drawings.
3. Explaining the ideas of surface area and curved surface area of solids by using different models made of wood or clay.

4. Using inductive - deductive approaches to teaching-learning.
5. Developing alternate approaches for solving problems.

4.3 Key Concepts

- Area of a triangle using Heron's formula.
- Surface area and volume of a cuboid and cube.
- Surface area and volume of a right circular cylinder.
- Surface area and volume of a right circular cone.
- Surface area and volume of a sphere.
- Surface area and volume of a combination of solids.
- Surface area and volume of a frustum of a cone.

4.4 Teaching Strategies

Now, we shall consider few of the key concepts from the above list for transacting in the classroom :

(i) Area of a triangle using Heron’s formula

(ii) Right circular cylinder and cone

(iii) Surface area and volume of a combination of solids

(iv) Surface area and volume of a frustum of a cone.

(i) Heron’s Formula

The teacher comes in the classroom and initiates the discussion as follows :

T : Do you know how to find the area of a triangle?

S : Sir! We can find area of a triangle as $\frac{1}{2}$(base) × (altitude or height).

T : Right. Suppose you do not know the altitude but instead you know only the three sides. Then what will you do?

S : We will draw the triangle and try to find its altitude and then calculate area using the same formula.

T : Find the area of a triangle with sides 5, 6 and 7 units by finding altitude corresponding to base 7.

S_1 : Sir! let us take altitude AD = p and BD = x.

S_2 : So, DC = 7 – x

T : Now, find relations in x and p.

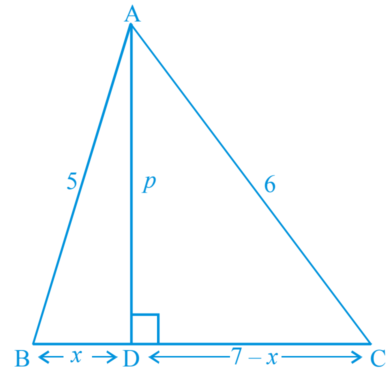


Fig. 1

S$_1$: Since Δ ADB is a right triangle, by Pythagoras theorem,

$p^2 + x^2 = 25$

S$_2$: Also, Δ ADC is a right triangle. So,

$p^2 + (7 - x)^2 = 36$

T : Equate the values of p^2. What do you get?

S : $p^2 = 25 - x^2 = 36 - (7 - x)^2$

or $25 - x^2 = 36 - 49 + 14x - x^2$

or $14x = 38$

or $x = \frac{19}{7}$.

T : Now calculate the value of p.

S : $p^2 = 25 - \frac{361}{49} = \frac{1225 - 361}{49} = \frac{864}{49} = \frac{6 \times 6 \times 6 \times 4}{49}$

or, $p = \frac{12}{7}\sqrt{6}$

T : Now find the area of the triangle using the formula.

S : Area of triangle $= \frac{1}{2}\text{base} \times \text{height}$

$= \frac{1}{2} \times 7 \times \frac{12}{7}\sqrt{6} = 6\sqrt{6}$.

T : Very good.

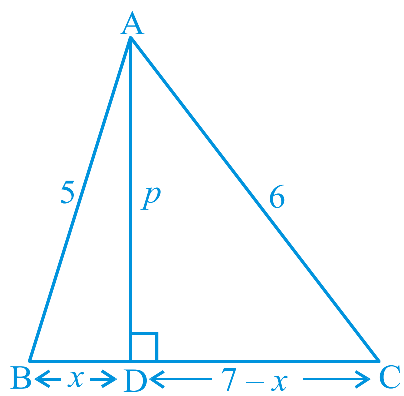


Fig. 2

T : Now find the area of a triangle having sides a, b and c units.

S$_1$: Let AD = p and BD = x as shown in the figure.

S$_2$: DC = $a - x$.

T : How will you proceed now?

S$_1$: Sir! ADB is a right triangle. So, $p^2 + x^2 = c^2$.

T : Then next ?

S_2 : Sir! ADC is also a right triangle. So,

$p^2 + (a - x)^2 = b^2$

T : Can you find the value of x, now?

S : Sir, by equating the two values of p^2, we have

$b^2 - (a - x)^2 = c^2 - x^2$

This gives $x = \dfrac{c^2 + a^2 - b^2}{2a}$

T : Right. Now find the value of p.

S : $p = \sqrt{c^2 - x^2} = \sqrt{c^2 - \left(\dfrac{c^2 + a^2 - b^2}{2a}\right)^2}$

T : Try to simplify it, using $2s = a + b + c$.

S : Sir! $p = \sqrt{\dfrac{(2ac)^2 - (c^2 + a^2 - b^2)^2}{(2a)^2}}$

$$= \frac{1}{2a}\sqrt{(2ac + c^2 + a^2 - b^2)(2ac - c^2 - a^2 + b^2)}$$

$$= \frac{1}{2a}\sqrt{\left[(c+a)^2 - b^2\right]\left[b^2 - (c-a)^2\right]}$$

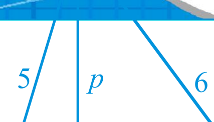
$$= \frac{1}{2a}\sqrt{(c+a+b)(c+a-b)(b+c-a)(b-c+a)}$$

$$= \frac{1}{2a}\sqrt{2s(2s-2b)(2s-2a)(2s-2c)}$$

$$= \frac{2}{a}\sqrt{s(s-b)(s-a)(s-c)}$$

T : Now find the area of Δ ABC by using the formula

$\text{Area} = \dfrac{1}{2}\text{Base} \times \text{Altitude}.$



S : Sir! area of Δ ABC

$$= \frac{1}{2} \times a \times p$$

$$= \frac{1}{2} \times a \times \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{s(s-a)(s-b)(s-c)}$$

T : Good. This formula is known as Heron's formula.

Thus, the area of a triangle with sides *a, b* and *c* is given by

area = $\sqrt{s(s-a)(s-b)(s-c)}$, where $s = \dfrac{a+b+c}{2}$.

T : Now find the area of triangle with sides 5, 6 and 7 units using Heron's formula.

S$_1$: Sir! here $s = \dfrac{5+6+7}{2} = 9$

S$_2$: Area = $\sqrt{9 \times (9-5) \times (9-6) \times (9-7)}$

$= \sqrt{9 \times 4 \times 3 \times 2} = 6\sqrt{6}$ square units

T: So you see that area comes out the same in both the cases.

Do you Know?

- Area of a cyclic quadrilateral having sides *a*, *b*, *c* and *d* is also given by a formula similar as to Heron's formula of a triangle.

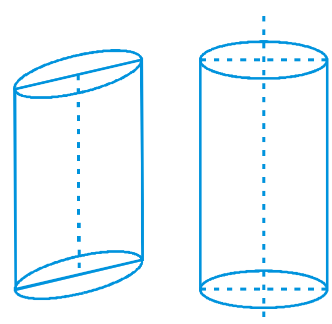
Area $= \sqrt{(s-a)(s-b)(s-c)(s-d)}$, where $2s = a+b+c+d$.

Review Questions

1. Write the formula for the area of an isosceles triangle with equal sides *a* and height h.
2. Write the formula for area of an equilateral triangle with side *a*.

Right Circular Cylinder and Cone

T : Well, look at the Fig 3, in which two cylinders are drawn on the blackboard. What is the difference between the two?



(i) (ii)

Fig. 3

S : In Fig 3(i), the axis of the cylinder is not perpendicular to its base while in Fig 3(ii), the axis is perpendicular to the base.

T : The cylinder in Fig 3(ii) is called a right circular cylinder but the cylinder in Fig 3(i) is not a right circular cylinder.

S : Sir! is it necessary that the base of a cylinder is always circular?

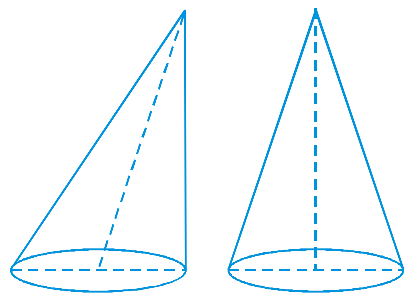
T : No. It can be elliptical or may be surrounded by any curve. [However it will not be easy to find the volume or surface area of such cylinders. Do you remember the formulae for finding volume and surface area of a right circular cylinder when its radius is r and height is h?]

S : Sir, they are given in the textbook as: volume of a right circular cylinder = $\pi r^2 h$
curved surface area = $2\pi rh$

Total surface area = $2\pi r(h + r)$

T : Like cylinder, cones can also be of two types. Look at the Fig 4 drawn on the blackboard. Can you tell from the figure, which one is right circular?

S : Sir! the cone in Fig 4(ii), is right circular because the axis is perpendicular to the base.



(i) **Fig. 4** (ii)

Review Question

In what way, a right circular cylinder is different from a cylinder?

(ii) Surface Area and Volume of Combination of Two Solids

T : You must have seen a toy like the Fig 5(i) and a solid like as in Fig 5(ii). Can you tell what solids are these?

S_1 : Sir, the base of Fig 5(i) is a hemisphere while its top is a cone.

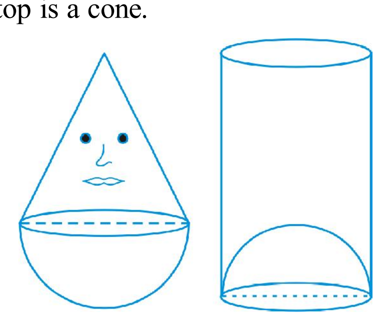
S_2 : The solid in Fig 5(ii) is like a cylinder from which a hemisphere has been hollowed out.

T : Absolutely right. Today, we will see how the volumes and surface areas of combination of two known solids can be found.

S : Sir! I can find the volume in both the cases.

T : How?

S : If volumes of two solids are V_1 and V_2 and they are joined, its volume will be $V_1 + V_2$. If a solid



(i) **Fig. 5** (ii)

of volume V_1 is taken out from a solid of volume V_2, the volume of remaining will be $V_2 - V_1$.

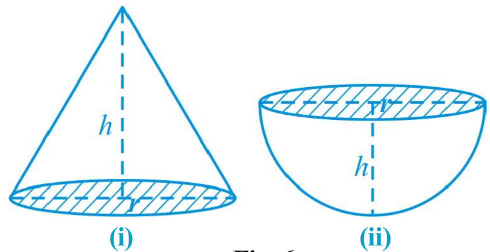


Fig. 6

T : Good.

S : Will the similar result be true for the surface area also?

T : Try to find the surface area of Fig 5(i) when radius of hemisphere or cone is r and height of cone is h.

S₁ : Surface area of cone [Fig. 5(i)] $= \pi r\left(\sqrt{h^2 + r^2} + r\right)$

S₂ : Surface area of hemisphere [Fig. 6 (ii)] $= 2\pi r^2 + \pi r^2 = 3\pi r^2$

S₃ : Sum of surface areas of two solids

$$= \pi r\left(\sqrt{h^2 + r^2} + r\right) + 3\pi r^2$$

$$= \pi r\left(\sqrt{h^2 + r^2} + 4r\right)$$

S₄ : Sir, total surface area (TSA) of the solid in Fig 5(i)

$$= \pi r(\sqrt{h^2 + r^2} + 4r)$$

T : Is it correct ?

S₁ : No, sir.

S₂ : Total Surface Area of toy [Fig 5 (i)]

= Curved Surface Area (CSA) of cone + Curved Surface Area (CSA) of hemisphere

$$= \pi r\left(\sqrt{h^2 + r^2}\right) + 2\pi r^2$$

$$= \pi r\left(\sqrt{h^2 + r^2} + 2r\right)$$

T : Alright. What did you find?

S : Sir, the surface area of the combination is less than the sum of surface areas of the two solids.

T : Do you know how it happened?

S : Sir, the base of cone of area πr^2 and base of hemisphere of area πr^2 overlap each other and are not visible and, therefore, the sum of these areas is not included in the surface area of the toy.

T : Now if a solid is cut out from another solid, can you guess what will happen to the surface area of the remainder, whether it will increase or decrease?

S_1 : Certainly it will decrease.

S_2 : It may increase also.

T : Both of you are right. It depends on the particular situation. Sometimes it increases and sometimes it decreases.

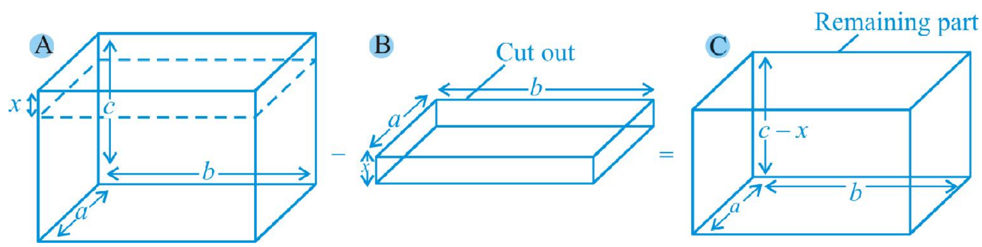


Fig. 7

S : Sir! please elaborate a little more. It is not clear to us.

T : First, let us take the case when it decreases. Look at the figure drawn on the black-board. (Fig. 7).

T : What is the surface area of solid A?

S : It is $2(ab + bc + ac)$.

T : What is surface area of solid C which is the remaining part?

S : It is $2[ab + a(c - x) + b(c - x)]$

$= 2\,(ab + bc + ac) - 2x(a + b)$

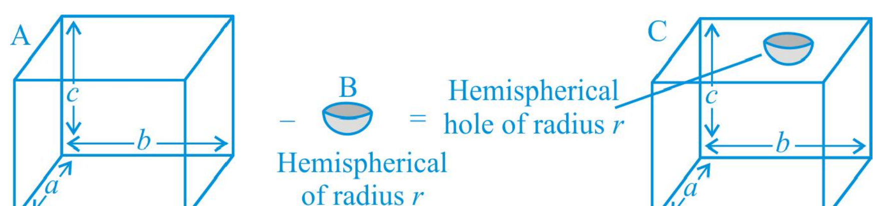


Fig. 8

T : So, it is less than surface area of solid A. Now let us take the case when surface area increases after cutting out a solid. Look at Fig.8 drawn on the black- board. Here a hemispherical hole is made along a face of a cuboid.

T : What is the surface area of solid A?

S : It is again $2(ab + bc + ac)$.

T : What is the surface area of solid C when a hemispherical hole of radius r is made?

S_1 : It will be $2(ab + bc + ac) + 2\pi r^2$

T : It is not correct. Can anybody correct it?

S_2 : Sir! it will be

$2(ab + bc + ac) + 2\pi r^2 - \pi r^2$

$= 2(ab + bc + ac) + \pi r^2$

T : Correct. So, you see that surface area has increased after cutting out a solid from the given solid.

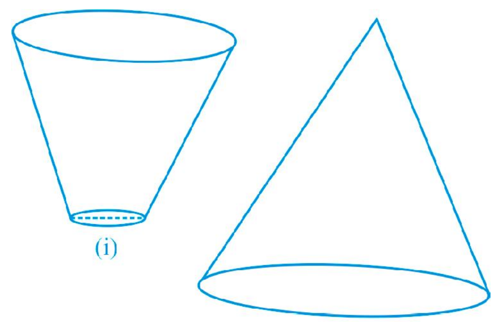


Fig. 9

Review Question

1. Give one or two examples of combination of solids:
   (i) Whose surface area is more than the sum of surface areas of solids from which it is made.
   (ii) Whose surface area is less than the sum of surface areas of solids from which it is made.
   (iii) Where surface area is equal to the sum of surface areas of solids from which it is made.

(iii) Frustum of a Cone

T : Let us now consider the solid given in Fig 9(i) which is like the shape of a glass used for drinking water. As you know this shape is found when a small cone is cut out from a bigger cone.

S : Sir, what is the name of this figure?

T : It is called **a frustum of cone.**

S : Suppose we cut a small piece from a cone like the figure (Fig. 9(ii)) will it again be a frustum?

T : No. The base of the cone cut out should be parallel to the base of given cone. Only then it will be a frustum. For finding volume and surface area, we must consider the frustum of a right circular cone only.

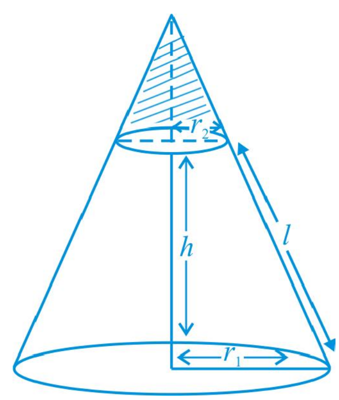


Fig. 10: Frustum of cone

T : Do you know the formulae for finding volume and surface area of a frustum of cone? [unshaded in Fig. 10].

S : Sir, these are given in the textbook as :

Volume $=\frac{1}{3}\pi h\left(r_1^2+r_1r_2+r_2^2\right)$

Curved surface area $=\pi l\left(r_1+r_2\right)$

Total surface area $=\pi l\left(r_1+r_2\right)+\pi r_1^2+\pi r_2^2$,

where r_1, r_2 are the radii of its ends, h is the height and l is the slant height of the frustum of cone.

S : Sir! how can we derive these formulae?

T : You must be knowing the properties of similar triangles. These will be used in deriving these formulae.

Let height AB of the frustum be h and r_1, r_2 be the radii of its ends and let CD = l be its slant height. Complete the cone. Let O be its vertex, OA = h' and OC = l' . (See Fig. 11)

Now Δ OAC is similar to Δ OBD. What will you get then?

S : $\frac{OA}{OB}=\frac{AC}{BD}=\frac{OC}{OD}$ or $\frac{h'}{h'+h}=\frac{r_2}{r_1}=\frac{l'}{l+l'}$

T : What are values of h¢ and l¢, then?

S : Sir, $h'=\frac{r_2}{r_1-r_2}h$ and l¢= $\frac{r_2}{r_1 - r_2}l$

T : Now, volume of frustum

= volume of bigger cone – volume of smaller cone

Remember that volume of a cone

$=\frac{1}{3}\pi(\text{radius})^2\times\text{height}$

So, what will be the volume of the frustum?

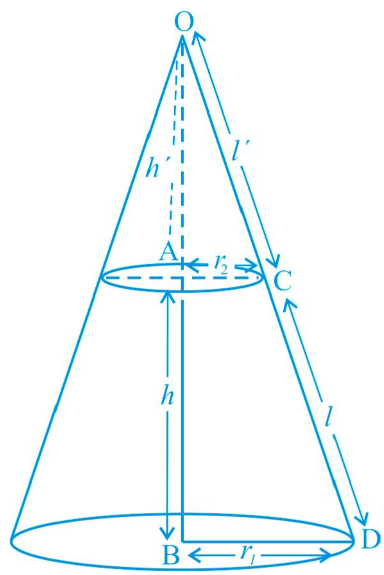


Fig. 11

S : Sir, volume of the frustum

$=\frac{1}{3}\pi r_1^2\left(h'+h\right)-\frac{1}{3}\pi r_2^2h'$

$$= \frac{1}{3}\pi r_1^2 h + \frac{1}{3}\pi\left(r_1^2 - r_2^2\right)h'$$

$$= \frac{1}{3}\pi r_1^2 h + \frac{1}{3}\pi\left(r_1^2 - r_2^2\right)\frac{r_2}{r_1 - r_2}h$$

$$= \frac{1}{3}\pi h\left[r_1^2 + \left(r_1 + r_2\right)r_2\right]$$

$$= \frac{1}{3}\pi h\left(r_1^2 + r_1 r_2 + r_2^2\right)$$

T **:** Now find the curved surface area of the frustum, in the same way.

S **:** Curved surface area of the frustum $= \pi r_1\left(l' + l\right) - \pi r_2 l'$

$$= \pi r_1 l + \pi\left(r_1 - r_2\right)l'$$

$$= \pi r_1 l + \pi\left(r_1 - r_2\right)\frac{r_2}{r_1 - r_2}l$$

$$= \pi\left(r_1 + r_2\right)l$$

T **:** What will be the total surface area?

S **:** Total surface area of frustum

$$= \pi\left(r_1 + r_2\right)l + \pi r_1^2 + \pi r_2^2 .$$

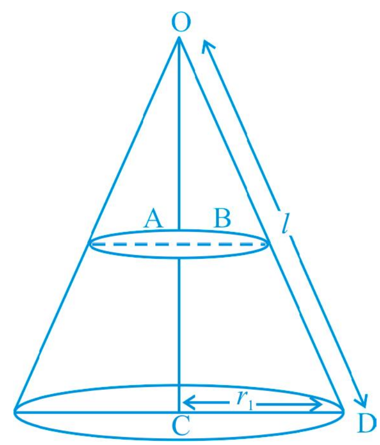


Fig. 12

Review Questions

1. Give two examples of objects whose shapes are like a frustum of cone.
2. As curved surface of a cylinder can be imagined to be formed by folding a rectangular sheet, what type of sheet will give a frustum of a cone on folding it?
3. Show that if slant height of a cone is divided by a plane parallel to the base in the ratio $\sqrt{2} + 1 : 1$ from the vertex, then its curved surface area (CSA) is divided into two equal parts.

Solution Q. 3.

Let base radius of the cone be r and its slant height be l (= OD) (Fig. 12). Then

$$\text{OB} = \frac{\sqrt{2}+1}{\sqrt{2}+2}l = \frac{l}{\sqrt{2}} \quad \text{and} \quad \text{AB} = \frac{r}{\sqrt{2}}$$

$$\text{CSA of small cone} = \pi \times \frac{r}{\sqrt{2}} \times \frac{l}{\sqrt{2}} = \frac{1}{2}\pi r l$$

$$= \frac{1}{2} \text{ C.S.A. of full cone.}$$

Alternatively, C.S.A. of frustum

$$= \pi\left(l - \frac{l}{\sqrt{2}}\right)\left(r + \frac{r}{\sqrt{2}}\right) = \pi r l\left(1 - \frac{1}{2}\right) = \frac{1}{2}\pi r l$$

$$= \frac{1}{2} \text{ CSA of full cone.}$$

Let us take an example.

Example: An oil funnel of tin sheet consists of a 10cm long cylindrical portion attached to a frustum of cone. If the total height is 22cm, diameter of the cylindrical portion is 8cm and the diameter of the top of funnel is 18cm, find the area of the tin sheet used in making the funnel (see Fig 13).

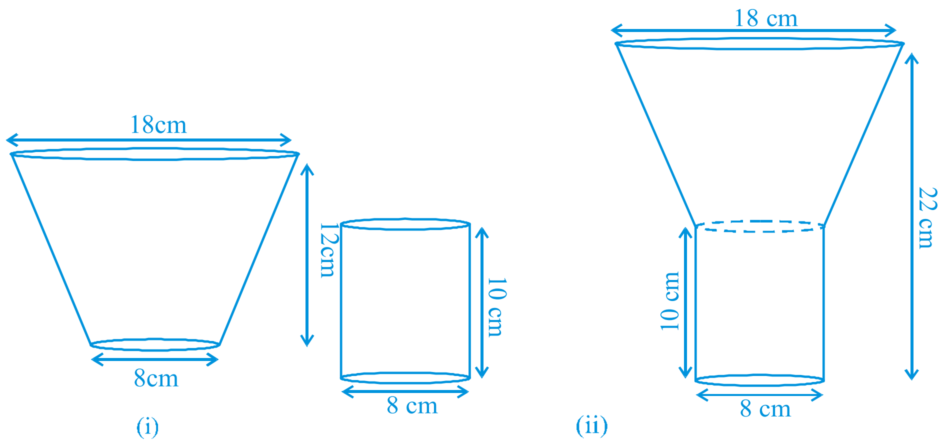


Fig. 13

The teacher draws the figure for the problem.

S : Sir, draw the two portions separately.

T : These can be shown as in Fig 13(i) and (ii).

To find area of the tin sheet, you have to find the curved surface area of the two portions separately and then add them.

S : Sir, how to find the slant height of the frustum of cone?

T : Look at the figure ABCD shown in Fig. 14.

Here AB is the line segment joining the centres of the two faces, CD is the slant height. So, AD and BC are the radii of the two faces. Draw CL perpendicular from C on AD.

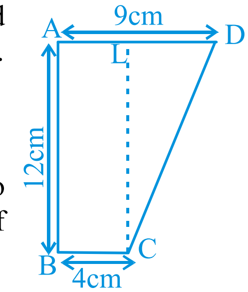


Fig. 14

S : Sir, D CLD will be a right triangle in which CL = 12 cm, LD = (9 – 4)cm = 5cm

So, CD $= \sqrt{5^2 + 12^2}\,\text{cm} = 13\text{cm}$.

T : Correct. Now you can find the required area.

S : Curved surface area of frustum $= \pi(r_1 + r_2)l = \frac{22}{7} \times (4+9) \times 13\,\text{cm}^2$

Curved surface area of cylinder $= 2\pi rh = 2 \times \frac{22}{7} \times 4 \times 10\,\text{cm}^2$

Total surface area $= \frac{22}{7} \times (169 + 80)\,\text{cm}^2 = \frac{22 \times 249}{7}\,\text{cm}^2$

$$= \frac{5478}{7}\,\text{cm}^2$$

$$= 782\frac{4}{7}\,\text{cm}^2$$

T : Yes, you got the answer.

4.5 Misconceptions

1. Unit of surface area is taken as unit of volume and vice-versa.
2. By taking out a solid from another solid, the surface area of the remaining part is taken as difference of the surface areas of the two.
3. By joining one solid to another, the surface area of the solid so formed is taken as sum of surface areas of the two solids.
4. Cylinders and cones are always right circular and with circular bases only.

Assignment for Teachers

1. Describe a method of estimating the surface area of a cylinder and write it in dialogue form.
2. A wall of length 10m, breadth 22 cm and height 154 cm is to be constructed with bricks of dimensions 22 cm × 11 cm × 7 cm. If material used to join the bricks occupies 1/10th of the total volumes, estimate the number of bricks to be used using dialogue form with students of the class.

Activities

1. Take a rectangular sheet of hard paper. Attach a straw or thin stick along one of its edges. Revolve or rotate about it through a full turn. What surface do you find? It will be a cylinder (right circular). (See figure 15)

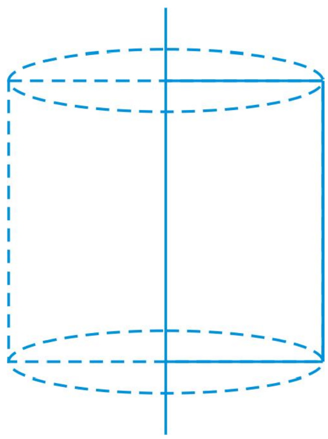


Fig. 15

2. Take a sheet of hard paper in the form of a right triangle. Attach a straw or a thin stick along one of its legs. Rotate the sheet about it through one full turn. The surface formed in this way will be a right circular cone (See figure 16).

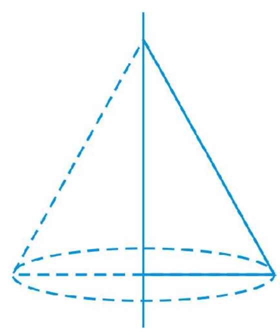


Fig. 16

3. Take a sheet of hard paper in the form of a trapezium whose one non-parallel side is perpendicular to parallel sides. Attach a thin stick along this side. Rotate the sheet about the stick through one full turn. The surface formed will be a frustum of cone (See figure 17).

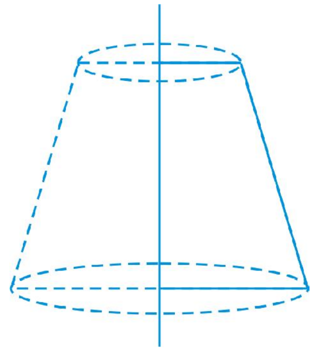


Fig. 17

4. Take a sheet of hard paper in the form of a semicircle. Attach a thin stick along its diameter. Rotate the sheet about this stick through a full turn. The surface thus formed will be a sphere (See figure 18).

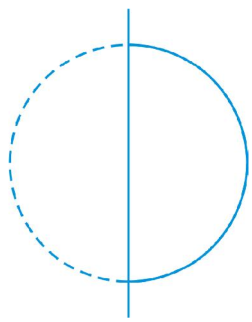


Fig. 18

Exercise

Tick (√) the correct options in questions 1 and 2:

1. If A_1 and A_2 are the areas of top and bottom of a frustum of cone of height h, its volume is

(A) $\frac{h}{3}(A_1+A_2)$ (B) $\frac{h}{3}\sqrt{A_1^2+A_2^2+A_1A_2}$

(C) $\frac{h}{3}\left(A_1+\sqrt{A_1A_2}+A_2\right)$ (D) $\frac{h}{3}\sqrt{A_1A_2}$

Ans. (C)

2. If two sides of a triangle are 6cm and 8cm and its area is 24cm^2, then third side is :

   (A) 10 cm (B) 7 cm (C) $4\sqrt{3}$ cm (D) $\frac{48}{7}$ cm

   Ans. (A)

Write true or false in Questions 3 and 4 and justify your answer.

3. The altitude corresponding to base 14cm in a triangle with sides 13cm, 14cm and 15cm is 12cm.

   (True)

4. A cone of radius 4cm is divided into two parts by cutting it through a plane parallel to the base and through the mid-point of the height. Then volumes of the two portions are in the ratio 1:5.

   (False : Ratio is 1:7)

5. A triangle has two sides measuring 17cm and 21cm. If its area is 84cm^2, find the third side.

   (Ans. 10 cm or $4\sqrt{85}$ cm)

6. The diameters of two circular ends of a frustum of cone are 8cm and 16cm. If its total surface area is 440cm^2, find its height.

   (Ans. 3cm)

UNIT 5

TEACHING OF STATISTICS AND PROBABILITY

5.1 Introduction

In day to day life, we come across a wide variety of information in the form of facts, numerical figures, tables, graphs etc., through newspapers, televisions, magazines and other means of communications.

These informations may relate to cost of living, cricket averages, profits/losses of a company, temperatures and rainfalls of cities, expenditures in various sectors of a 5–year plan, polling results, etc.

These facts or figures which are numerical or otherwise collected with a definite purpose, are called **data**. Every facet of life uses/utilises data in one form or the other. So, it is essential to know how to extract meaningful information from such data. This extraction of information is studied in a branch of mathematics called **statistics**.

The word **statistics** appears to have been derived from the Latin word 'status ' meaning a '(political)' state.

In olden days, the government used to collect information regarding the population, property or wealth of the country in order to have an idea of man power of the country, possible imposition of new taxes/levies and design future development plans.

In India, an efficient system of collecting data existed even more than 2000 years ago during the reign of Chandragupta Maurya (324–300 BC). From Kautilya's **Arthshastra,** it is evident that even before 300 BC, a very good system of collecting **vital statistics** and registration of deaths was in vogue. During Akbar's reign (1556–1605), Raja Todarmal, the then Land and Revenue Minister, maintained good records of law and agricultural data. In **Ain-i-Akbari** written by Abul-Fazal one of the nine gems of Akbar in 1596–97, a detailed account of the administrative and statistical surveys conducted during that period can be found.

The 17th century saw the origin of **vital statistics**. Captain John Graunt of London (1620–1674), known as the 'Father of Vital Statistics' was the first person to study statistics of births and deaths. Computation of mortality tables and the calculation of life expectancy at different ages led to the idea of 'life insurance'.

With the passage of time, the scope of statistics was **widened** and it began to include collection of numerical data from almost every sphere of life such as imports, exports, marriages and divorces, and presentation of data in tabular and pictorial forms. By the end of nineteenth century, the scope of statistics was further widened to include collection, organisation, presentation and interpretation of data leading to meaningful inferences.

These days, computers help us in processing data at a very fast rate and making predictions. Our society uses **statistics** to analyse trends and making decisions. Economists analyse financial trends, psychologists measure intelligence, newspapers/TV's ascertain our opinions on a certain issue/problem, educators assess achievements of students, weather department analyse data available in the past to make future predictions etc.

5.2 General Strategies

- Motivation to study statistics by citing examples from day to day life. The students should be encouraged to collect data themselves from their immediate environment and use them in learning other concepts related to statistics such as tabulation, graphical representation, analysis and interpretation.
- As far as possible each new concept may be introduced through examples taken from different situations.

5.3 Key Concepts

- The word 'statistics' used in singular and plural senses.
- Data : meaning of data – collection of data, primary and secondary data, raw/ungrouped data.
- Presentation of data in the form of a table – frequency distribution table.
- Graphical representation of data
  - Bar graphs
  - Construction and interpretation
  - Histograms of equal class width and of varying class widths
  - Frequency polygons
- Measures of central tendency
  - Mean, mode, median of ungrouped data

  - Mean, mode, median of grouped data
  - Ogives – less than type, more than type
- Probability
  - experimental
  - theoretical

5.4 Teaching Strategies

Among the key concepts listed above, we now discuss a few :

(i) Teaching of histograms

(ii) Teaching of measures of central tendency — median for a continuous grouped frequency distribution.

(iii) Teaching of probability – experimental and theoretical

(i) Teaching of Histograms

Before teaching of histograms, the teacher has to ascertain that the students are well versed with the following concepts which form prerequisites for learning histograms.

- Meaning of data
- Graphical representation of data – pictograph, bar graph – drawing of a bar graph
- Grouped frequency distribution.
- Area of a rectangle
- Ratio and Proportion

Teacher may now initiate discussion in the following way :

T : Well students, you are already aware of a bar graph. Today, we shall discuss another form of graphical representation of data which also looks like a bar graph but it is certainly different from a bar graph.

Teacher writes the following frequency distribution on the blackboard:

Marks	Number of students
0-10	5
10-20	10
20-30	8
30-40	5
40-50	2

and asks students what information does this table give?

S : Sir, this table gives information about the number of marks obtained by 30 students in a test.

T : What information does first column and what information does second column give?

S : First column lists the range of marks obtained and second column lists the number of students, that is, frequency of the class.

T : What type of distribution is it?

S_1 : It is a grouped frequency distribution.

S_2 : It is continuous.

T : Good. It is a continuous grouped frequency distribution. Today, we shall learn how to represent such type of distribution graphically. How should we proceed?

S : We draw two mutually prependicular lines intersecting at a point O.

Teacher asks now from other students, 'What to do next?'

S : We represent marks along the horizontal axis or x-axis.

T : How?

S : By taking a suitable scale. For example, we may take 1cm to represent 10 marks.

T : Anuj, how to go further?

S : Sir, we may write 'marks $\rightarrow$' below horizontal axis.

S : We may take number of students (frequency) along vertical axis or y-axis.

T : How?

S : Again taking a suitable scale.

S : Since the maximum frequency is 10, we have to choose a scale to accomodate this maximum frequency.

S : Sir, we can take 1 cm to represent 2 students.

T : Should we also label this axis? and how?

S : Sir, yes.

S : We should label it as 'Number of students' or 'frequency'$\rightarrow$.

> Teacher supervises that each student is involved in the process and following the construction of "histogram" stepwise.

T : You already know bar graph. How to proceed further?

S : We can erect rectangles or bars as we did in the case of drawing a bar graph.

S : Sir, on class interval 0-10, should we draw a rectangle of length (height) 5?

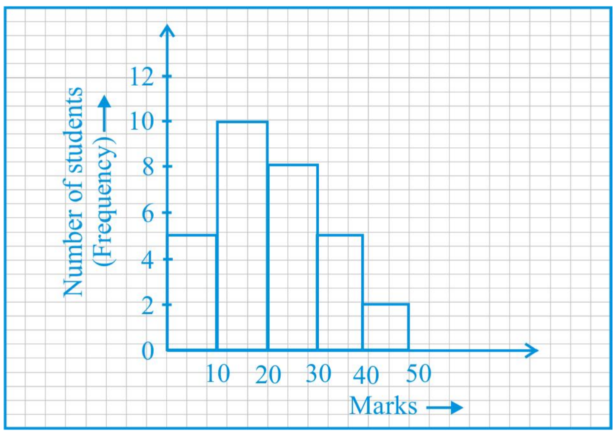
T : Yes, draw a rectangle of width equal to the class size and length according to the frequency of the class.

Teacher helps in drawing one rectangle on the class interval 0-10 and asks the students to follow the same procedure for drawing rectangles on other class intervals.

T : Have you drawn all the five rectangles?

S : Yes, sir.

T : Apoorv, come forward and draw the figure on the blackboard.



Apoorv draws the figure on the blackboard.

T : This graph represents the given data. It is a **histogram**.

S : Sir, what is the meaning of word histogram?

T : Here 'histo' means area and 'gram' means graph. In what way, is it different from a bar graph?

S_1 : Sir, it is also a bar graph.

S_2 : I see that there are no gaps in between rectangles, as used to be in a bar graph.

S_3 : Sir, it appears to be a solid figure.

T : Well! You have observed a few differences between a histogram and a bar graph. What is the width of each class here?

S : It is 10.

T : Yes, each class is of width 10, i.e., class width is 10. What is the breadth of each rectangle?

S : Sir, it is also 10.

Now teacher explains that **areas of rectangles are proportional to the corresponding frequencies** by asking the students to find area of each rectangle.

T : What is the area of each rectangle?

S_1 : Area of rectangle on width (class) 0-10 is $10 \times 5 = 50$ square units.

S_2 : Area of rectangle on width 10-20 is $10 \times 10 = 100$ square units.

S_3 : Area of rectangle on width 20-30 is $10 \times 8 = 80$ square units

S_4 : Area of rectangle on width 30 – 40 is $10 \times 5 = 50$ square units.

S_5 : Area of rectangle on width 40 – 50 is $10 \times 2 = 20$ square units.

T : Students! Can you find any relationship between area of a rectangle and frequency of the respective class?

S_1 : For first rectangle,

area = class width × height of rectangle (frequency)

= 10 × frequency = 10 × corresponding frequency

S_2 : For second rectangle

area = 10 × corresponding frequency

S_3 : Same is the case for other rectangles.

T : So, can we say that area of each rectangle is *proportional* to its frequency?

S : Yes, Sir.

T : We can write :

area of a rectangle $\propto$ frequency

i.e., $\boxed{\text{area of a rectangle} = k \times \text{frequency},}$ where k is a constant of proportionality.

What is constant of proportionality in this case?

S : It is 10 here.

T : Dear students! Today, we have learnt how to represent a continuous grouped frequency distribution graphically. This representation is called a histogram.

Teacher now will give 2-3 more continuous grouped frequency distributions to the students and ask them to draw their histograms following the steps as discussed above.

T : Well students!

Here is again a grouped frequency distribution :

Class	Frequency
10–24	1
25–39	4
40–54	10
55–69	6
70–84	3
85–99	4

Draw a histogram to represent the distribution.

S_1 : Sir, in the previous case, the classes were continuous. But in this case, it is not so.

S_2 : So, we cannot draw its histogram.

S_3 : Sir, if we make the data continuous, then it is possible to draw the histogram.

T : Good! You got the point. Make the classes continuous.

S : Sir, I can make the classes continuous as

9.5 – 24.5
24.5 – 39.5
39.5 – 54.5
54.5 – 69.5
69.5 – 84.5
84.5 – 99.5

S_2 : So, we have now the following table:

Class	Frequency
9.5 – 24.5	1
24.5 – 39.5	4
39.5 – 54.5	10
54.5 – 69.5	6
69.5 – 84.5	3
84.5 – 99.5	4

T : Can you now construct a histogram for this data?

S : Yes sir.

T : What are the class widths now?

S : 24.5 – 9.5 = 15, 39.5 – 24.5 = 15
54.5 – 39.5 = 15, ..., 99.5 – 84.5 = 15,
i.e., 15 in each case.

T : How will you draw a histogram?

S_1 : Since the class widths are same, we will follow the same procedure as followed earlier for continuous grouped frequency distribution.

S_2 : Here, we shall take classes –
9.5 – 24.5, 24.5 – 39.5, ..., 84.5 – 99.5
along x-axis and respective frequencies along y-axis
and then draw rectangles as we did in the last example.

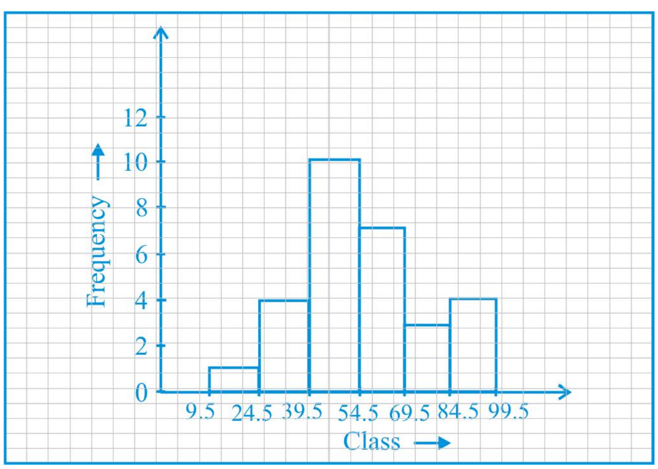
Students are now busy in drawing rectangles on respective classes as breadths and taking heights of rectangles as frequencies.

T : Have you drawn the histogram?

S : Yes, Sir.

T : Anita, come here on the blackboard and draw the histogram.

Anita draws histogram



T : What is the area of the rectangle made on class 9.5 – 24.5 as its breadth?

S_1 : Sir, area of this rectangle

= 15 × height of the rectangle (frequency).

= 15 × frequency = 15 × 1

T : What is the area of the rectangle made on 24.5 – 39.5?

S_2 : It is 15 × height of the rectangle

i.e., 15 × frequency = 15 × 4 = 60

T : What is the area of rectangle made on 39.5 – 54.6?

S_3 : The area of this rectangle

= 15 × height of the rectangle

= 15 × frequency = 15 × 10

T : Well students!

You have found that area of each rectangle

= 15 × frequency of its class

This means:

Area of each rectangle is *proportional to its frequency*

or, Area of each rectangle α frequency

Avin, tell, what is constant of proportionality here?

Avin : Is it not 15, Sir?

T : Yes. Check that it is the same for each rectangle.

Teacher now reinforces that in a histogram, there is no gap between rectangles and areas of rectangles are proportional to their respective frequencies.

Teacher may give 2-3 grouped frequency distributions of the type discussed above (i.e., with classes not continuous but of equal widths) and asks students to draw their histograms. Also, asks students to find constant of proportionality in each case.

T : Students! You already know how to represent a grouped frequency distribution graphically using a histogram. Let us draw a histogram of the data :

Mass (in kg)	Frequency
6 – 9	4
9 – 12	6
12 – 18	10
18 – 21	3
21 – 30	12

S$_1$: Sir, I can draw a histogram of this data.

T : How will you draw?

S$_2$: We first draw two perpendicular lines meeting at a point O.

T : What next?

S$_1$: We take classes along horizontal axis, i.e., *x*-axis and respective frequencies along vertical axis, i.e., *y*-axis, choosing an appropriate scale.

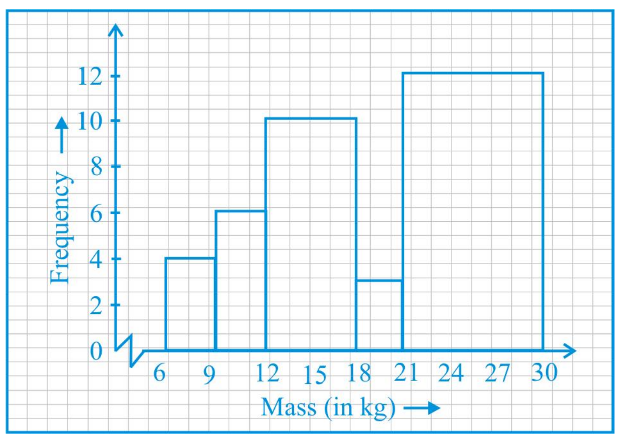
S$_2$: Then, we make rectangles on each class interval with their heights equal to respective frequencies.

The students are now busy in drawing their histograms.

T : I think, now you have completed the work. Who will come forward to draw it on the black board so that all of us can also see it?

S : Sir, I can do it.

Teacher asks her to draw a histogram of the given data. Student draws it as follows:



T : Is it okay?

S : Yes, Sir.

T : Let us find area of each rectangle. What is the area of the rectangle drawn on class width of 6 – 9?

S : It is 3 × height

= 3 × frequency

= 3 × 4 = 12 square units

T : What is the area of next rectangle?

S : Area of this rectangle = 3 × height = 3 × frequency = 3 × 6 = 18 square units.

T : What is the area of next rectangle drawn on class width of 12 – 18 ?

S : It is 6 × height = 6 × frequency = 6 × 10 = 60 square units

T : What are the areas of rectangles drawn on class widths of 18 – 21 and 21 – 30 ?

S : They are : 3 × height = **3 × frequency** = 3 × 3 = 9 square units

and 9 × height = **9 × frequency** = 9 × 12 = 108 square units

T : As you know, in a histogram, area of each rectangle is proportional to its frequency, what is the constant of proportionality in each case here?

S_1 : In case of first rectangle, it is 3

S : In case of 2nd rectangle, it is 3

S : In case of 3rd rectangle, it is 6

S : In case of 4th rectangle, it is 3

S : In case of 5th rectangle, it is 9

T : Is the constant of proportionality same?

S : It is same for 3 rectangles but not for other two.

T : So, this histogram is **not correct**.

Students are discussing among themselves, why it is not correct. Teacher, then asks the students whether width of each class is the same.

S : No. It is the same in case of 6 – 9, 9 – 12 and 18 – 21, i.e., 3 but not the same for 12 – 18 (which is 6) and for 21 – 30 (which is 9).

T : Since classes are not of the same width, so areas of the rectangles are not proporational to frequencies.

Recall that in previous example, widths of classes were the same. So were the widths of the rectangles.

Therefore, areas of rectangles were proportional to their frequencies.

S : Sir, what to do in this case?

T : Here, we have to 'adjust' frequencies in such a way that areas of rectangles are proportional to their frequencies. In such a case, we take one of them as **standard** frequency and determine the other frequencies using the formula :

$$\text{Adjusted frequency} = \frac{\text{standard frequency}}{\text{class size}} \times \text{its frequency}$$

Suppose, we take standard frequency as 3.

Then,

adjusted frequency of class 6 – 9 is $\frac{3}{3} \times 4 = 4$

adjusted frequency of class 9 – 12 is $\frac{3}{3} \times 6 = 6$

adjusted frequency of class 12 – 18 is $\frac{3}{6} \times 10 = 5$

ans so on.

What is the adjusted frequency of class 18 – 21?

S : Sir, it will be $\frac{3}{3} \times 3 = 3$

T : What is the adjusted frequency of class 21 – 30?

S : Sir, adjusted frequecny = $\frac{3}{9} \times 12 = 4$

T : Now, rewrite the table using these adjusted frequencies.

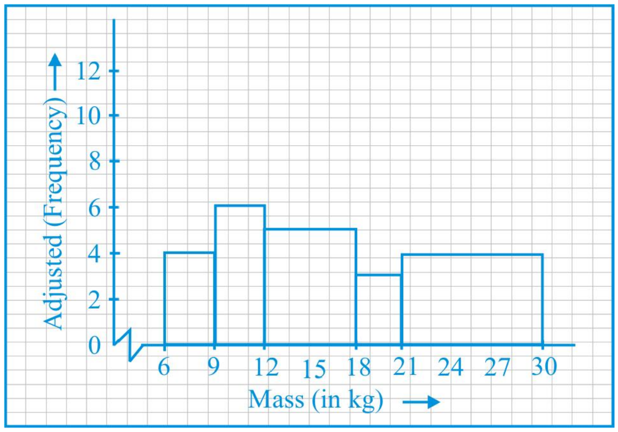
S :

Mass(in kg)	Adjusted frequency
6 – 9	4
9 – 12	6
12 – 18	5
18 – 21	3
21 – 30	4

T : Now, you may draw a histogram of this 'new' distribution.

Students now, draw histogram accordingly.

T : Shobha, come here and draw a histogram of this distribution.



T : Yes, it is a correct histogram.

Well students now, you can check that areas of rectangles are proportional to the frequencies.

Can you check it?

S : Area of rectangle on class width of 6 – 9 = 3 × 4

Area of rectangle on class width of 9 – 12 = 3 × 6

Area of rectangle on class width of 12 – 18 = 6 × 5 = 30 = 3 × 10

Area of rectangle on class width of 18 – 21 = 3 × 3

Area of rectangle on class width of 21 – 31 = 9 × 4 = 3 × 12

T : So, area of each rectangle = 3 × its frequency

Hence, area of each rectangle α its adjusted frequency

and the constant of proportionality = 3

Teacher now advises the students that in case of classes of varying widths, you may take any class width as **standard frequency** and asks the students to do the same question by taking standard frequency as 6 or 9 or any other. But to minimise computation and avoid fractional frequencies, take only that which gives **adjusted frequencies** as whole numbers.

Review Questions

1. Discuss the construction of a frequency polygon for a grouped frequency distribution with the students in a similar way as discussed above.

2. Discuss the construction of cumulative frequency curves (ogives) both 'of less than type' and 'of more than type' for a given cumulative frequency distribution, in a similar way as discussed for 'histograms'.

(ii) Median of a Grouped Frequency Distribution

Before teaching of median for a continuous grouped distribution, the teacher has to ascertain that the students are well versed with the following concepts which are pre-requisite to study the following:

- Median – one of the measures of central tendency
- Median is that value of the observation which divides data into two equal (nearly) parts when data are arranged in ascending (or descending) order.
- Median of raw data
- Cumulative frequency distribution
- Calculation of median of ungrouped data

Teacher may start discussion in the following way :

T : Students! You have studied about mean of the data — both for ungrouped and grouped data which is one measure of central tendency. Median is another measure of central tendency. You have already studied how to find median of ungrouped data. Today, we shall discuss how to find median of continuous grouped frequency distribution.

For example, let us find median of the following distribution :

Ages (in years)	Number of Teachers (frequency)
20 – 25	7
25 – 30	11
30 – 35	18
35 – 40	20
40 – 45	15
45 – 50	14
50 – 55	15

Can you recall how to find median of ungrouped data.

S : Sir, in case of ungrouped data, we first prepared a cumulative frequency table. Should we not do it here also?

T : Yes

S : So, we should first prepare a cumulative frequency table.

T : Okay, prepare it.

Ages (in years)	Frequency	Cumulative Frequency
20 – 25	7	7
25 – 30	11	18 (7 +11)
30 – 35	18	36 (7 + 11 + 18)
35 – 40	20	56
40 – 45	15	71
45 – 50	14	85
50 – 55	15	100

T : Okay. It is a cumulative frequency table. How many observations are in all?.

S : 100, Sir.

T : Let it be n, i.e., $n = 100$.
Which observation will be the median?

S : Here n is even (i.e, 100). So median will be average (mean) of $\left(\frac{n}{2}\right)^{th}$ and $\left(\frac{n}{2}+1\right)^{th}$ observations.

T : So, what are $\left(\frac{n}{2}\right)^{th}$ and $\left(\frac{n}{2}+1\right)^{th}$?

S : $\frac{n}{2} = 50$ th, $\left(\frac{n}{2}+1\right) = 51$ th

T : In which class, these two observations lie?

S : They, both lie in the class 35 – 40.

T : This class 35 – 40 is called **median class** as both the observations (50^{th} and 51^{st}) lie in this class. But we do not know which are exactly 50^{th} and 51^{st} observations.

S_1: How to proceed further?

S_2: Sir, one thing is clear that median will be 35 or more.

S_3: And less than 40.

T : How to proceed further?

S : I do not know. Please help me.

T : To overcome this difficulty, we use the following formula for finding median in such situations :

$$\text{Median } = l + \left(\frac{\frac{n}{2} - cf}{f} \right) \times h,$$

where l = lower limit of median class

n = number of observations

cf = cumulative frequency of the class preceding the median class

f = frequency of median class

h = class size (assuming all classes are of equal width).

So what is l here?

S_1: Median class is 35 – 40. Its lower limit is 35.

So, $l = 35$

S_2: Here $n = 100$ and $\frac{n}{2} = 50$

T : What is 'cf' here?

S : It is cumulative frequency of the class preceding the median class.

T : What is the median class?

S : 35 – 40

T : Which class is just before the median class?

S : 30 – 35

T : Then what is cf of it?

S : It is 18?

T : Is it correct?

S : No, Sir. 18 is the frequency of 30 – 35.

T : Then, what is its cf ?

S : It is 36, i.e., $cf = 36$

T : Yes, it is correct now.

What is 'f' here?

S : f is the frequency of median class.

T : Then f is equal to _______ ?

S : It is 20.

T : Yes, $f = 20$

What is 'h' here?

S : $h = 5$ here.

T : How?

S : $25 - 20 = 30 - 25 = 35 - 30 = 40 - 35 = 45 - 40 = 50 - 45 = 55 - 50 = 5$

Each class has width (size) 5.

T : Now calculate median of the given distribution by putting these values of l, $\frac{n}{2}$, cf, f and h in the formula.

Now students are busy in calculating the median.

After some time,

T : Have you calculated the median?

S : Sir, I have done it.

S : Sir, I am about to complete it.

T : Okay. Aslam, have you calculated the median?

Aslam: Sir, it is 38.5

T : Write your calculations on the blackboard.

(Aslam writes on the black board) Median $= l + \left(\frac{\frac{n}{2} - cf}{f}\right) \times h$

$$= 35 + \left(\frac{50 - 36}{20}\right) \times 5$$

$$= 35 + \frac{14 \times 5}{20}$$

$$= 35 + 3.5 = 38.5$$

T : Yes, your answer is correct, i.e.,

Median of the data $= 38.5$

or median age of teacher is 38.5 years.

S : What does it show, Sir?

T : Median age 38.5 years shows that about 50 % teachers have their ages less than 38.5 years and 50 % have their ages more than 38.5 years.

T : By convention in the formula, we take either $\frac{n}{2}$ or $\frac{n+1}{2}$ depending on the convenience of performing calculations.

S : Sir, this is not clear to us. How has this formula come?

T : Do you recall that to calculate mean of grouped data, we made an assumption that observations in a class are centered at its mid-point (class mark). A similar type of assumption, we make here for developing the formula. The assumption is "Observations in a class are equally spaced or distributed."

S : Please explain it. It is not clear to us.

T : You see that there are 20 observations in the median class 35 – 40. Its size is 5. If we divide the length of this class interval, i.e., 5 into 20 equal parts, each part will have value $\frac{5}{20}$.

Number of observations upto median observation in the median class 35-40 = 14 (i.e., (50 – 36))

So, value of 14. i.e., (50-36) parts $= (50-36) \times \frac{5}{20}$

So, $\text{Median} = 35 + \left(\frac{50-36}{20}\right) \times 5$

$$= l + \left(\frac{\frac{n}{2} - cf}{f}\right) \times h$$

50th obervation is 14 parts away from the lower limit of the median class.

So, value of 50th observation is $= 35 + \frac{5}{20} \times 14 = 38.5$

Even, if you calculate value of 51st observation, it will be $= 35 + \frac{5}{20} \times 15 = 38.75$

So, median $= \dfrac{38.5 + 38.75}{2} = \dfrac{77.25}{2} = 38.6$ years (approx.)

The difference of both 38.5 and 38.6 is **just** 0.1 which is negligible.

S : Sir, here again, suppose the classes are not continuous. Then how to proceed in that case.

T : Then, you have to make the classes continuous.

Review Questions

1. Discuss calculation of mode for continuous grouped frequency distribution with the students in a similar way as discussed for median.
2. Discuss calculation of mean for grouped frequency distribution with the students in a similar way.

(iii) Probability

T : Students!

Tell me, will it rain in the evening today?

S_1: Sir, it **may** rain, as it is very hot today.

S_2: Sir, it **may** rain or it **may not** rain.

Yesterday, it was hot in the morning, but it did not rain in the evening.

T : It means, you cannot give a **definite answer** to my question.

Now, see what is it?

S : It is a drawing pin.

T : If I drop it, will it land point down?

S : It is **unlikely** to land point down.

T : How can you say that?

S_1: Generally, when it is dropped, it falls on either the top or side.

S_2: Sir, it can also land point down.

T : Here again, you are still not sure whether it will land point down or not. So, there is no **definite answer** to my question.

You know, tomorrow, there is a cricket match between India and Pakistan. Who will win the toss?

S_1: India

S_2: Pakistan

S_3: It may be India or Pakistan. We cannot say definitely who will win the toss.

T : Again, there is no definite answer to my question.

Will the prices of petrol products go up next month?

S_1: Sir, may be!

S_2: Sir, prices may go down also.

S_3: It is also possible that prices may neither go up nor go down. These may remain as they are now.

T : It means, there is again no definite answer to my question.

So, dear students, there are some questions, as I posed, to which it is not possible to **give a definite answer**. Each of these questions involve the word may, may not etc. There is an **element of uncertainty** in answering each of these questions.

This uncertainty can be measured numerically which is studied under the topic called *probability*.

Probability started by the inquiries of gamblers to win stakes at games related to tossing of coins, rolling of a die, playing cards etc., it has been used extensively in many fields of physical sciences, commerce, biological sciences, medical sciences, insurances, weather forecastings etc.

This is head This is tail

T : Take a one rupee coin. Each of you, toss it 20 times and note down the number of heads and tails.

Each student does this activity, and teacher writes number of heads and tails on the blackboard and so on for all the students.

Student	Number of heads	Number of tails
1	6	14
2	12	8
3	9	11
.		
.		
40	–	–

T : How many tosses?

S : $20 \times 40 = 800$

T : How many times did you get heads?

S : It is 480.

T : How many times did you get tails?

S : 320

T : We may call each toss of a coin a **trial**. So, there are 800 trials, in which heads are 480 and tails are 320.

T : Here is a die. Have you seen a die before?

S_1: I have seen it.

S_2: We used a die in playing ludo.

S_3: We used it in playing 'snake and ladder' game also.

T : What more do you know about a die?

S_1: It is cubical in shape.

S_2: Numbers 1, 2, 3, 4, 5, 6 are written on its faces

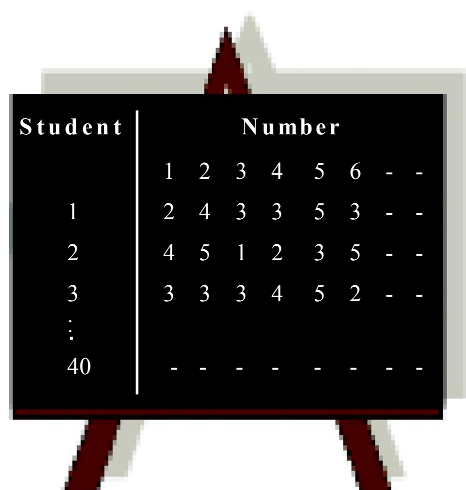
S_3: I have seen dots, ·, ··, ∴, ∷, ⁙, ⠿ on its faces.

T : Have you ever seen that sum of numbers/dots on opposite faces of a die is 7?

S : No, Sir. Let us check it in this die.

Sir, it is true.

T : Now, every one of you will throw it 20 times and tell me the number that appears on the top of it.



Student	Number							
	1	2	3	4	5	6	-	-
1	2	4	3	3	5	3	-	-
2	4	5	1	2	3	5	-	-
3	3	3	3	4	5	2	-	-
⋮								
40	-	-	-	-	-	-	-	-

So, each student does this experiment and teacher makes a table on the blackboard.

T : Count number of times '1' has appeared '2' has appeared, '3' has appeared, ... '6' has appeared.

How many trials?

S_1: Sir, $20 \times 40 = 800$ trials.

S_2: Number of times '1' appeared is 132

S_3: '2' appeared : 148 times

S_4: '3' appeared : 120 times

S_5: '4' appeared : 164 times

S_6: '5' appeared : 124 times

S_7: '6' appeared : 112 times

T : Head (also written as H) and Tail (T) are two **outcomes** of the **experiment** of tossing a coin.

In case of throwing a die, what are the **outcomes**?

S_1: 1, 2, 3.

S_2: 1, 2, 3, 4, 5, 6

T : In tossing a coin, 'getting a head' is an event. If we toss a coin and head comes, we say that event 'getting a head' has occurred. Similarly, 'getting a tail' is also an event. In case of throwing a die, 'getting 1' is an event, 'getting 2' is also an event.

What about 'getting 3'?

S_1: This is also an event.

S_2: 'getting 4,' 'getting 5,' 'getting 6' are also events of the experiment of throwing a die.

T : We denote an event by 'E'.

Can you give any other event associated with the experiment of throwing a die?

S : 'getting an odd number,' i.e., 1, 3, 5

S : 'getting an even number,' i.e., 2, 4, 6

S : 'getting a number more than 2,' i.e., 3, 4, 5, 6

T : So, there may be many events associated with the experiment of throwing a die.

We define (experimental) probability of an event E [denoted by $P(E)$] as

$$P(E) = \frac{\text{Number of trials in which the event has occurred}}{\text{Total number of trials}}$$

S : Sir, why we call it experimental probability?

T : As we have done an experiment, that is why we call *P(E)* as **experimental** probability.

It is also called emperical probability but here we shall use the word **probability** for experimental probability.

Thus, if E is the event 'getting a head,' then for the experiment of tossing a coin

Probability of event E, $P(E) = \dfrac{480}{800} \begin{array}{l} \leftarrow (\text{Number of times Heads appeared}). \\ \leftarrow (\text{Total number of trials}) \end{array}$

i.e., probability of getting a Head = $\boxed{0.6}$

What is probability of getting a Tail?

S : Probability of 'getting a tail' = $P(E) = \dfrac{320}{800} = 0.4$

T : In the experiment of throwing a die, what is the number of trials?

S : 800

T : What is the probability of the event 'getting 1'?

S : It is $\dfrac{132}{800}$.

S : If E denotes event 'getting 1',

$$P(E) = \frac{132}{800} = \boxed{0.165}$$

T : What is the probability of 'getting 2'?

S : $P(\text{getting } 2) = \dfrac{148}{800} = \boxed{0.185}$

T : What is the probability of 'getting 3'?

S : $P\ (\text{getting } 3) = \dfrac{120}{800} = \boxed{0.150}$

S : Now, I can tell the probability of 'getting 4', 'getting 5', 'getting 6'.

T : Yes.

S : $P\ (\text{getting } 4) = \dfrac{164}{800} = \boxed{0.205}$

$P\ (\text{getting } 5) = \dfrac{124}{800} = \boxed{0.155}$

$P\ (\text{getting } 6) = \dfrac{112}{800} = \boxed{0.140}$

T : Find the sum of all these probabilities, i.e., P(getting 1) + P(getting 2) + ... + P(getting 6).

S : Sir, $0.165 + 0.185 + 0.150 + 0.205 + 0.155 + 0.140 = 1$

T : In the experiment of tossing a coin, what is the sum of P(getting head) = P(H) and P(getting T) = P(T))?

S : $P(H) + P(T) = 0.6 + 0.4 = 1$.

T : So, we can say that the sum of probabilities of all the outcomes of an experiment is 1. What else do you see about each of these probabilities?

S : Each one is a fraction less than 1.

S : Each one is a fraction greater than 0.

T : So, P(E) is a number lying between 0 and 1.

In the experiment of throwing a die,

What is the probability of "getting 7".

S : In this experiment, 7 does not appear anytime.

S : In fact, it cannot appear, as 7 is not written on either of faces of the die.

S : So, $P(\text{ getting } 7) = \frac{0}{800} = 0$

T : Such an event which cannot happen is called **an impossible event**. Probability of an 'impossible event' is 0.

What is the probability of the event "getting a number 1, 2, 3, 4, 5 or 6"?

S : Naturally, one of them will always appear.

S : So, $P(1, 2, 3, 4, 5 \text{ or } 6) = \frac{800}{800} = 1$

T : Such an event which is almost certain to happen is called **a sure event** and probability of a'sure event' is 1.

Thus,

Probability of an event E = P(E) is a number such that $0 \le P(E) \le 1$.

Review Question

1. Discuss introduction of theoretical probability in a similar way as discussed for introduction of experimental probability.

Note:

1. A grouped frequency distribution may have a class with frequency 0 (zero). In such case, for drawing a histogram, the rectangle erected on that class will have length (height) 0, i.e., the rectangle will be a line segment along the horizontal line representing that class.
2. For finding 'adjusted frequencies' in case of drawing a histogram having classes of varying widths, it is not necessary that we should take the minimum width of a class as **standard** frequency. In fact, it could be any width of the given classes.
3. Mode is defined as the observation with highest frequency not the largest observation.
4. In using 'assumed mean method' for finding mean of grouped frequency distribution, it is not necessary to take the middle most x_i as assumed mean. In fact, it could be any other x_i or any other number. But it is easy to calculate mean, if we take middle most x_i as assumed mean.
5. Step-deviation method for finding mean of grouped frequency distribution is applicable only when all classes are of the same width.
6. In the textbook, calculation of median has been discussed from a ‘less than type ogive’. Median can also be calculated from ‘more than type ogive’ in a similar way.
7. Mean of a given raw data and mean calculated from the data converted in the form of a grouped frequency table may be different. But mean of a frequency distribution calculated using either assumed mean method or direct method or step deviation method will be the same.
8. Under certain conditions, a relationship

   3 Median = Mode + 2 Mean

   holds between median, mode and mean. The students should not be encouraged to use this relationship in finding one of the three when other two are given. For more detail, refer to some standard book on statistics.

5.5 Misconceptions

1. For the data involving classes of varying widths, sometimes, students draw histogram as in the case of classes of uniform width without finding ‘adjusted frequencies’. In such cases, “adjusted frequencies” have to be used in place of given frequencies.
2. Some students may draw a histogram of the following data by having an impression that the first column has continuous classes and 2nd column has frequencies.

Year	Production (in tonnes)
2000 – 01	350
2001 – 02	410
2002 – 03	720
2003 – 04	550
2004 – 05	300

But this is not so. In fact, it is not a grouped frequency distribution. However, in such cases, we may draw a bar graph.

3. Some students think that arithmetic average or mean is the only average. It is not so. Median and mode are also averages and there are some more averages also, which they will study in higher classes.

4. Some students think that relationship, 3 Median = Mode + 2 Mean is always true. In fact, it is true only under some specific conditions.

5.6 Exercise

1. A survey or recorded the number of people living in each of 40 houses as follows :

3	4	5	2	2	3	4	2	4	3
2	5	4	5	6	4	2	3	2	4
4	1	2	6	3	5	2	4	1	5
3	4	2	6	4	4	2	4	3	4

(i) Complete the following table :

Number of people	Tally marks	Number of houses
1		
2		
3		
4		
5		
6		

(ii) Draw a bar graph for the data in table above.

2. Draw a histogram to represent the data :

Height (in cm)	Frequency
130–135	6
135–140	12
140–145	18
145–150	14
150–155	0
155–160	8
Total	**58**

[Hint **:** Height of rectangle erected on 150–155 will be 0. It means that rectangle is a line segment]

3. Draw a histogram to represent the following distribution :

Length (in cm)	Frequency
26 – 30	4
31 – 35	10
36 – 40	18
41 – 45	13
46 – 50	5
Total	50

4. The table below shows the ages of teachers in a school :

Age (in years)	20–30	30–45	35–40	40–45	45–60
Frequency	10	18	24	9	15

Draw a histogram to represent the data.

5. The heights (in cm) of plants were measured during an experiment and the results are summarised as in the table below :

Height (in cm)	Frequency
0 – 5	20
5 – 10	40
10 – 15	35
15 – 20	25
20 – 25	50

(i) Draw a cumulative frequency table for this distribution

(ii) Which class interval contains the median height?

(iii) Determine the median height.

6. Find the mean and mode of the distribution in Q. 5.

7. Find median of the following distribution :

Marks	Frequency
6 – 15	4
16 – 25	14
26 – 35	18
36 – 45	5
46 – 55	8
56 – 65	4
66 – 75	2
Total	**55**

8. Find the mean of the data in Q. 7

9. Find the mode of the data in Q. 7

UNIT
6

PROBLEM SOLVING IN MATHEMATICS

6.1 Introduction

Mathematics plays a very important role in our day-to-day life. Mathematical skills and its applications form an indispensable tool in our lives, the world over. It is said, 'Higher the achievement in the field of mathematics, so is the nation's development'. Much of human's progress in the last few centuries has made it necessary to apply mathematics to problems of varied nature which were previously coming under the perview of customs and traditions. These days Mathematics is being used more and more in Social Sciences, Management and Commerce in addition to Basic and Life Sciences. Problem solving has, therefore, acquired an important place in the teaching-learning process of mathematics. Besides other approaches, problem solving is also considered as an important approach for teaching mathematics.

6.1.1 What is a Problem?

In general, a problem may be explained as description of an objective not immediately attainable, calling for an action appropriate to attainment of that objective. For example, if someone is hungry, then objective is to satisfy the hunger and appropriate action is to look for food at a place where there is a possibility of getting it. Suppose a person wants to succeed in a competitive examination, then appropriate actions for the attainment of the objective are :

(i) to look for the syllabus on which the examination is based.

(ii) to analyse the type of questions that are set in the examination.

(iii) to prepare for the examination in the light of the above information etc.

According to G. Polya year (1981, P.117), a problem is a situation in which a person searches consciously for some appropriate action to attain the clearly, but not easily attainable objective and solving a problem means to find such an action.

In mathematics also, problem is a situation for which the child does not have an immediate answer or an obvious mathematical operation or method of finding the answer. On the other hand, problem solving may be considered as a process by which the child discovers some previously known rules that she can apply to achieve a solution for that problem. Here it must be noted that a problem can be solved in a number of ways and the teacher must encourage the child to follow her own approach(s).

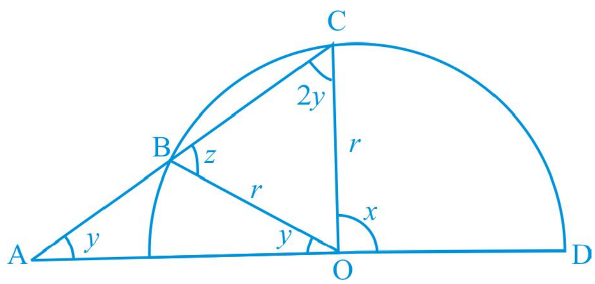
6.2 Importance of Problems and Problem Solving in Mathematics

You must have heard of some of famous mathematical problems which have puzzled a number of mathematicians in past. For example :

1. **Trisecting an angle using ruler and compasses**

   In this connection, it must be remembered that 'no markings' are allowed on the ruler. For this, consider the following problem :

   In the adjoining figure, OB = AB, where B is the point of intersection of the line segment AC and a semicircle drawn with O as centre and OC as radius. Show that $\angle y = \frac{1}{3}\angle x$.



Solution:

As AB = OB, so $\angle BOA = \angle y$.

Therefore, $\angle z = 2\angle y$

So, $\angle BCO = \angle z = 2\angle y$ (OB = OC)

Therefore, $\angle x = \angle y + 2\angle y = 3\angle y$

or, $\angle y = \frac{1}{3}\angle x$

This is known as the *famous Archimedes problem.* With this, he claimed that he could trisect a given angle, if he is allowed to have two markings on the ruler.

2. **Squaring a Circle**
3. **Doubling a Cube**
4. **Euclid's Fifth Postulate:** In attempting to find a proof for it, mathematicians were able to discover Non-Euclidean geometries.
5. **The Prime Number Mysteries:** Some of the oldest unsolved problems involving prime numbers is the prime number mysteries. For example no one has been able to

write a formula or system that will provide a prime number next to a given prime number. There may be some way of forming prime numbers, but no one has yet been able to find a systematic way to do it.

Another mystery about prime numbers is asked by the question, 'Is there an infinite number of twin primes?' A twin prime is a pair of prime numbers whose difference is 2; for example (3, 5), (11,13), (41,43). These twin primes seem to occur throughout our number system. No one has been able to find how many there are or to discover a formula to determine them. But, on the other hand, no one has been able to prove that there is a number beyond which there are no twin primes.

6. **Goldbach's Conjecture:** 'Is every even number the sum of two primes?' This is another mathematical mystery. In 1742, the German mathematician C. Goldbach wrote a letter to his friend, the great Swiss mathematician Leonhard Euler (1707-1783), in which he made the conjecture that every even number except 2 was the sum of two primes. This was an interesting statement that was true for every even number he examined, but he could not prove that it was true statement for all even numbers.

   If you try some even numbers you will find that it always works; for example, 4 = 2+2, 6 = 3+3, 8 = 3+5. No even number has been found that is not the sum of two primes. But there is no proof that every even number is sum of two primes. If you could find one even number that is not the sum of two prime numbers, then the problem would be solved. Since no logical proof has been found for this seemingly simple problem, it is still one of the mysteries of mathematics.

7. **The Odd Perfect Number Mystery:** The ancient Greeks considered some numbers to be perfect. Perfect numbers are numbers which are equal to the sum of their divisors. The number 6 is such a number because 6 = 1+2+3. Another perfect number is 28, since 28 = 1+2+4+7+14. The next perfect number after 28 is 496. Others have been found and all of them are even numbers. No one has ever found an odd perfect number. Also no one has been able to prove that every perfect number must be even.

8. **How to Pack Spheres:** A geometry problem that is still unsolved involves the packing of spheres such as ping-pong balls. How should spheres be packed in a box so that they use the least possible space? This is similar to a problem of drawing circles in a rectangle.

9. **The Four Colour Map Problem:** There are also unsolved problems in the field of topology. One is the four colour map problem. How many different colours are needed to make a map so that countries with a common border are coloured differently? This is a real mystery to map makers and to mathematicians. They have not been able to draw a map that needs more than four colours. But at the same time they

have not been able to prove that four colours are enough for any possible map. However, technological proof is available for this problem which has been assumed as a mathematical proof.

Every day mathematicians and scientists are working on problems that seem to be unsolvable. The answer to some of these may be that the solution is impossible. To others, the answers will be new ideas that will open new worlds of mathematics. May be, you will be the one to become famous by finding the solution to one of these problems.

6.3 Classification of Problems

Before attempting to solve a problem, the child must be asked to observe the type of the problem she is going to solve and choose the appropriate action to solve it. In mathematics, generally problems will of be the following type:

(i) To simplify

(ii) To verify

(iii) To find

(iv) To form an equation and solve it

(v) To prove (or show) some statement.

Children should be encouraged to discuss problems and analyse them by asking the following questions :

(i) What information is given?

(ii) What is to be done?

(iii) Do you need any information which is not given?

(iv) Is there any information in the problem which is not to be used?

(v) What is to be done to solve the problem?

(vi) What are the facts known to you for solving the problem?

(vii) Do you need to draw a diagram to solve the problem?

(viii) Do you need to translate problem into the form of an equation or inequation for solving it?

(ix) Can you estimate the answer and check the reasonability of the solution obtained?

The above thinking helps in solving the problem, but it does not form a part of the solution. To explain it further, you may take some examples to illustrate the proper process of problem solving. However, it must be noted that this process is time consuming and sometimes we may fail in our attempts.

6.3.1 Some Examples for Problem Solving Techniques

Problem 1 : M is any point on the minor arc BC of a circumcircle of an equilateral triangle ABC. Prove that AM = BM + CM.

Problem 2 : The length of a rectangular strip is twice its width. How to cut it in three pieces, so that they can be rearranged in the form of a square?

Problem 3 : Factorise the algebraic expression $x^6 + 5x^3 + 8$.

Problem 4 : A King arranged his gold coins in the form of a square and left the palace asking the security guard not to disturb those coins. On his return, the King found that there were only two coins in the palace. The guard told him that three thieves attacked him and have taken all the gold coins leaving the two coins, because they could not be shared equally among those three thieves. The king punished the guard for telling a lie. Check, how the King came to know that guard was telling a lie.

Problem 5 : Prove that the line segment joining the mid-point of the hypotenuse of right triangle to its opposite vertex is half of the hypotenuse.

Problem 6 : The side of a triangle are of lengths 13 cm, 14 cm and 17 cm. The centre of the circle touching the two sides 13 cm and 14 cm lies on the longest side. Find the radius of the circle.

Problem 7 : A person entered an orchard having four gates and picked some apples. On leaving the orchard, he gave half of the apples and one more to the guard at the first gate, half of the remaining and one more to the guard at the second gate and so on up to the last gate. In the end, he was left with only one apple. How many apples did he gather in all?

Problem 8 : Construct a triangle when the lengths of its three medians are given.

6.3.2 Problem Solving and Creativity

By now, you must have realised that how interesting and creative is the process of problem solving. You must have heard that Pythagoras theorem can be proved in several ways. Similarly, try to collect such similar problems which can be solved in several ways. As a teacher, you must allow the child to solve and commit mistakes in solving a problem, so that the more permanent will be her learning.

Hints: Problems 1-8

Problem 1 : To prove that AM = BM + CM

Construction : Extend CM to N, so that

BM = MN. Join BN.

Proof. Let $\angle BAM = \theta$. Then

$\angle MBC = \angle MAC = 60° - \theta$

Therefore, $\angle ABM = 60° + 60° - \theta$

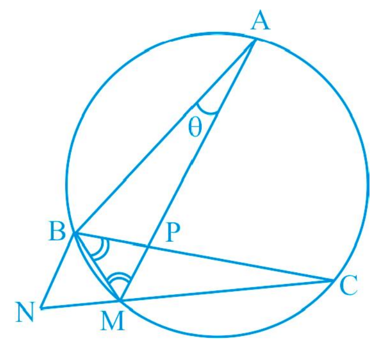
$= 120° - \theta$

Also, $\angle AMC = \angle ABC = 60°$

$\angle AMB = \angle ACB = 60°$

Therefore, $\angle BMN = 180° - (60° - 60°)$

$= 60°$



As BM = NM, Δ BMN is equilateral, i.e., MB = NB.

In Δ ABM and Δ CBN, AB=BC, BM=BN and $\angle ABM = 120°$

Therefore, $\Delta ABM \cong \Delta CBN$ (SAS)

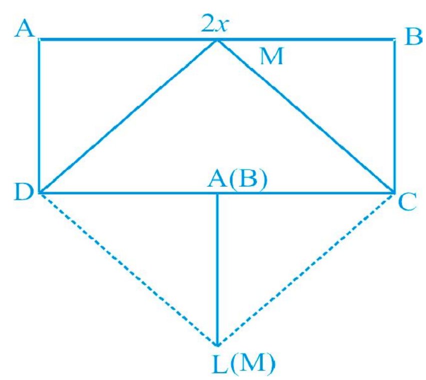
Therefore, AM = CN = CM + MN = CM + BM. (Also see unit 3)

Problem 2 : Area of rectangle = $x \times 2x$

Side of square $= \sqrt{2x^2} = x\sqrt{2}$

Join MC and MD. So MC = MD = $x\sqrt{2}$

and $\angle DMC = 90°$. This suggests to cut rectangle along MC and MD at three places and put these triangles as shown in adjoining figure.



Problem 3 : $x^6 + 5x^3 + 8$

$$x^6 + 8 + 5x^3 = \left(x^2+2\right)^3 - 6x^4 - 12x^2 + 5x^3$$

$$= \left[\left(x^2+2\right)^3 - x^3\right] - 6x^2\left(x^2 - x + 2\right)$$

$$= \left(x^2+2-x\right)\left[\left(x^2+2\right)^2 + x^2 + x\left(x^2+2\right)\right] - 6x^2\left(x^2 - x + 2\right)$$

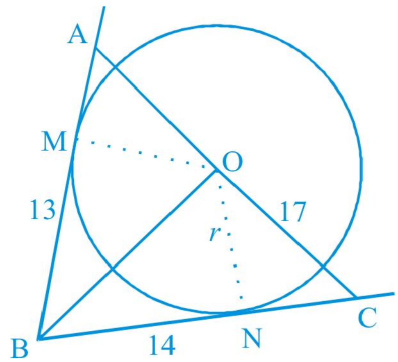
$$= \left(x^2+2-x\right)\left(x^4 + x^3 + 5x^2 + 2x + 4 - 6x^2\right)$$

$$= \left(x^2+2-x\right)\left(x^4 + x^3 - x^2 + 2x + 4\right)$$

Problem 4 : For any integer n, n^2 when divided by 3, the remainder is either 0 or 1, but not 2.

Problem 5 : Standard theorem.

Problem 6 : Join BO.



$$\text{Area of } \Delta\,\text{ABO} = \frac{1}{2} \times 13 \times r\,\text{cm}^2$$

$$\text{Area of } \Delta\,\text{BCO} = \frac{1}{2} \times 14 \times r\,\text{cm}^2$$

$$\text{So, area of } \Delta\,\text{ABC} = \frac{1}{2} \times 27 \times r\,\text{cm}^2$$

$$\text{Area of } \Delta\,\text{ABC} = \sqrt{22 \times 9 \times 8 \times 5}$$

$$= 12\sqrt{55}\,\text{cm}^2$$

$$\text{Therefore, } r = \frac{2}{27} \times 12\sqrt{55} = \frac{8}{9}\sqrt{55}\,\text{cm}$$

Problem 7 : Let number of apples be x.

Then after crossing the 1st gate, apples remaining $= \frac{x}{2} - 1 = \frac{x-2}{2}$

” ” 2nd ” ” $= \dfrac{\frac{x-2}{2}}{2} - 1 = \dfrac{x-6}{4}$

” ” 3rd ” ” $= \dfrac{\frac{x-6}{4}}{2} - 1 = \dfrac{x-14}{8}$

” ” 4th ” ” $= \dfrac{\frac{x-14}{8}}{2} - 1$ or $\dfrac{x-30}{16} = 1$

Therefore, $x = 30 + 16 = 46$

> If number of gates be n and apples remaining at the end be 1, then $x = 2^{n+1} + 2^n - 2$

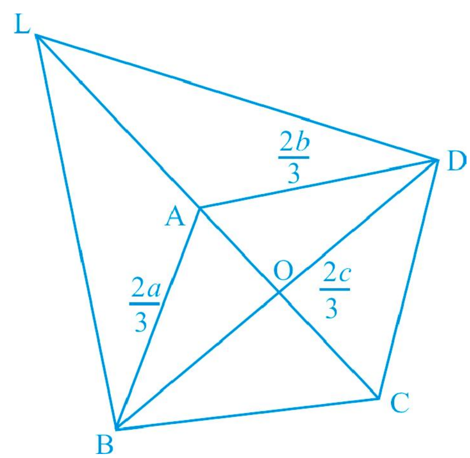
Problem 8 : Construction

Let lengths of medians be a, b and c.

Draw a parallelogram having two sides AB, AD of lengths $\frac{2a}{3}$, $\frac{2b}{3}$ and a diagonal AC $\frac{2c}{3}$ respectively.

Make AL = 2AO.

Then, Δ LBD is the required triangle.



UNIT 7

CONCEPT OF EDUCATIONAL EVALUATION

7.1 Introduction

Educational evaluation is of immense importance in the total span of educational process due to its potential of creating impact on the society as a whole. The merit of any educational system depends on the standards of attainment shown by its products in terms of competence and excellence. This shows that the quality of education is directly linked with the quality of evaluation. The scenario pertaining to evaluation system at different stages of schooling in the country presents a dismal picture as it ails with many shortcomings. Different Education Commissions and Committees reflected on these issues and suggested a number of ways to remedy these shortcomings. However, not much could be accomplished in this direction so far. One of the reasons attributed to this state of affair is lack in the conceptual understanding and procedural know-how regarding evaluation. This has led to carrying out evaluation in haphazard way in both internal and external examinations and created a mess instead of serving the desired purpose. Besides, there are many other lacunae in external examination which are hazardous and, therefore, retard the growth and quality of learning. It is necessary to strengthen the component of evaluation in order to upscale the quality of education.

7.2 Concept of Educational Evaluation

Generally speaking, evaluation is a very comprehensive term which includes evaluating any object, individual, institution, position of an office, event, trend, etc. However, educational evaluation deals with students' evaluation which includes the assessment of the performance of the students in the areas of their personality development in terms of intellectual, social and emotional development after they have been provided learning experiences through classroom processes. Besides the factors like quality of teaching, curricular material, instructional technology and school infrastructure, the societal support also influence his/her learning and enriches his/her experiences.

Evaluation is often confused with the term measurement and both terms are used synonymously. But both are not the same. The term measurement stands for measuring the performance of the student at a particular scale. The pattern of measurement which is mostly followed in our assessment system relates to marking on a scale of 0 - 100 marks. This also includes pass-fail system wherein all those who secure certain percent marks and above are declared pass and below this are tagged fail. This scale is a yardstick for classifying the students on the basis of the marks they obtain in a test or examination. Therefore, measurement provides a quantitative description of pupils performance based on artificial classification. It does not include value judgment and thus it gives a fragmented picture of student's performance. Moreover, all these aspects are related only to intellectual growth.

On the other hand, evaluation is a broader term as compared to measurement and it includes both quantitative and qualitative description of the performance and value judgment. Regarding quantitative description as discussed earlier, measurement on a scale is applied and marks are allotted. For qualitative description, interpretation of the marks secured by the student is made in reference to him/herself, his/her group and certain criteria. Evaluation also includes value judgment regarding the desirability of behaviour related to all the domains of personality development. The relationship of evaluation and measurement by Gronlund (1981) has been given as :

Measurement – quantitative description

Evaluation – quantitative description (measurement) (and/or)

– qualitative description (non-measurement) plus value judgment.

Thus, evaluation may or may not be based on definite measurement and goes beyond the simple quantitative score.

Evaluation has also been defined by various educationists as follows :

According to Tiwo (1995), the term "evaluation involves both quantitative and qualitative description of events, behaviours, things, parameters or variables as well as value judgment of things or events being described."

R.W. Tyler (1950) defined evaluation "as a systematic process of determining the extent to which educational objectives are achieved by pupils."

The most extended definition of evaluation has been supplied by C.E. Beeby (1977) who described evaluation as "the systematic collection and interpretation of evidence of learning as part of the process to a judgment of value with a view to action."

On the basis of above definitions, it may be concluded that evaluation is a process of collecting evidences about student's achievement or development, in terms of educational objectives. Judgments are formed and decisions are taken on the basis of evidences. Evaluation, therefore, has the following four components :

- information gathering
- interpretation of information
- judgment forming, and
- decision making

Information gathering pertains to the collection of evidences regarding pupil's performance in specific subject through responses to oral questions, during the classroom interaction, observing the pupils' interaction with other group members in group activities, giving tests and scoring the answer scripts. Regarding social and personal qualities, evidences can be collected through observing the behaviour indicators related to the identified qualities. As far as the performance in co-curricular areas are concerned such as in drawing, dance, drama and music, etc., the evidences can be captured through both observations and written tests.

Information gathering is followed by analysis of the evidences and forming the judgment regarding the pace of learning as well as level of the learning of the pupils. Analysis of evidences is done in terms of three reference points. The first point is related to the previous performance of the learner, whether he/she has improved or deteriorated in terms of his/her achievement? The second point is his/her standing with reference to the performance of his/her peer group constituting the whole class. The third point is concerned with the criteria determined by the teacher, whether the learner could attempt successfully all the given questions or he/she could do only a part of it. Analysis leads to decide whether the learning has effectively taken place.

Decision making is the next step of evaluation process. On the basis of the judgment, decision can be taken in the form of allotting marks or grades. Decision may be communicated to the pupils and parents through report card or certificates, etc.

Some Other Related Terms

There are some related terms associated to evaluation which need to be understood otherwise misunderstandings are carried on. Following related terms are explained as under :

Examination

It is a process of collecting evidences about pupil's achievement at the end of a time interval after the learning process has taken place. Therefore, an examination includes developing a number of tests, conducting them and then marking answer scripts or awarding grades for reporting the achievements of the students.

Test

Test is a tool consisting of a number of questions for finding out the knowledge, understanding, aptitude and interest, etc. of the students. The test is based on a pre-determined set of objectives.

Assessment

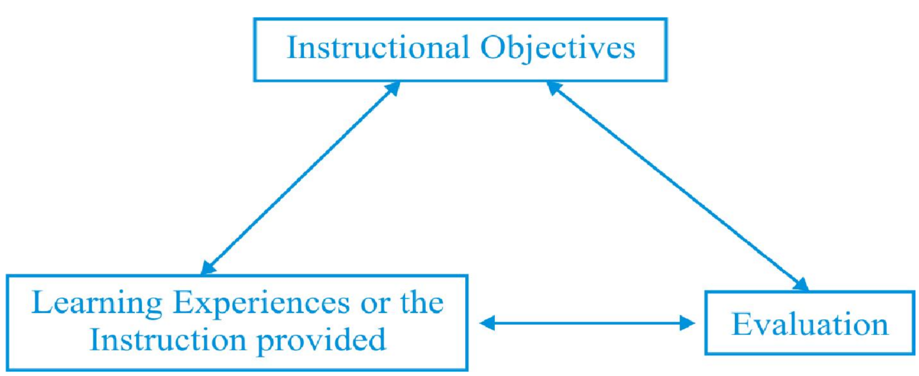
Assessment is the process of estimating the status of pupils development in different aspects of learning. e.g. knowledge, skills attitude etc. for the purpose of elevating further performances and learning outcomes. Therefore, assessment is followed by providing suggestions for improvement of performance. It can be done both in terms of quality or quantity. In British literature, assessment is occasionally used as a synonym to American term evaluation (Navo-1995).

7.3 Evaluation in Teaching-Learning

Evaluation is the term used to describe the determination of the level of quality of current performances. Evaluation process focuses only on the actual level of quality with no interest in why that level was attained. Evaluation report mentions only the level of quality and its possible consequences. It is not used to provide suggestions for further improvement. Although assessment and evaluation have different purpose, similar steps are involved in both the processess.

Evaluation is an integral part of any teaching-learning process. Whenever a question is asked in a class and answered by a student and the answer is judged by the teacher, evaluation takes place. Thus, both teaching and evaluation go hand in hand with each other. In fact, it is not possible to have teaching-learning without evaluation.

Both teaching and evaluation are based on the instructional objectives which provide direction to them. Instructional objectives are those desirable behaviour which are to be developed in students through the learning experiences. These are reflected in the form of syllabus, instructional material and the learning activities arranged by the teacher. Learning activities are carried out for achieving the objectives and evaluation is done to see whether the instructional objectives have been achieved and to what extent. The interrelationship of objectives, instructional process or the learning experiences and evaluation in a programme of teaching can be expressed more clearly through the following diagram :



The above diagram illustrates that the three components of teaching, learning and evaluation constitute an integrated network in which each component depends on the other. Thus, through evaluation, the teacher not only assesses as to how far the student has achieved the objectives, but also examines the effectiveness of the teaching strategy such as methodologies, means and the materials used for achieving those objectives.

7.4 Types of Evaluation

Evaluation is a quantitative as well as quantitative description, of all round development of the pupil including his physical, social, moral and intellectual development as well as his/her skills, abilities, aptitude and interests. Hence educational evaluation is a continuous, comprehensive, all-inclusive, and spatially-non-confined process addressing overall improvement.

Keeping wideness and enhanced scope of the concept, evaluation can by classified in several ways under various categories. Criteria for classification is the backbone for every classification system and hence, the evaluation too should follow some basis for classification. Types of evaluation can be given as:

(a) On the basis of purpose

(i) Diagnostic Evaluation

(ii) Prognostic Evaluation

(iii) Predictive Evaluation

(b) On the basis of continuity

(i) Formative Evaluation

(ii) Summative Evaluation

(iii) Comprehensive and Continuous Evaluation

(c) On the basis of reference point

(i) Criterion-Referenced Evaluation

(ii) Norm-Referenced Evaluation

A few of these categories are explained below:

7.4.1 Placement Evaluation/Entry behaviour

Placement evaluation is meant for finding out the position of the child in the initial stage of learning. When a teacher tries to introduce some learning experiences to the children in a class, it is imperative to know where a child can be placed in terms of his previous knowledge, so as to enable him to be ready for the further learning. This placement is a very important stage of evaluation, as it gives the teacher an idea of the weakness and strength of the child's learning. If it is not ascertained and the learning experiences are provided, the child

probably will not be able to develop the new concepts because he/she does not have the sound background of pre-requisite learning. The child will be able to attain the desired abilities and competence through new instructional process only with a strong background of pre-requisites, otherwise his weakness will continue to persist and his/her attainment will be adversely affected. Therefore, to avoid this situation and to enable the child to be strong in the attainment of concept and competence, placement evaluation is very essential.

7.4.2 Formative Evaluation

Formative evaluation is in-built in the process of teaching-learning. It is desirable to know whether a student has developed a certain ability stated in the objective as well as it is also required to know about his/her progress during the course of teaching and learning. If there is any deficiency in his/her learning that can be removed by alternate strategies of teaching. This form of evaluation is known as *formative evaluation*. The main purpose of this evaluation is to find out the extent to which the child is following the instructional process. It provides feedback to both the teacher and the student regarding the progress of the student and the effectiveness of the teaching methods, so that teaching-learning process may be improved. This type of evaluation can be done by means of oral tests, observations, unit tests, informal class tests, assignments and other classroom activities. This evaluation is *continuous* in nature.

7.4.3 Diagnostic Evaluation

Diagnostic evaluation as the name itself suggests, is meant for diagnostic purposes. It enables in finding out the learning difficulties of a child in a particular subject with reference to conceptual understanding, process of learning, language deficiency etc. Sometimes formal testing helps in diagnosing the hard spots of learning but sometimes specific tests are prepared with a definite purpose of diagnosing the learning problems. In mathematics, the problem in learning may be due to lack of understanding of mathematical concepts, generalisations, processes and recognition of symbols where the children generally mistakes. The teacher is supposed to go deep into the problem through these tests and find out the specific difficulty of the child in learning a concept or a particular step in solving a problem. While carrying out formative evaluation, the diagnostic test supplements the process of evaluation. If diagnosis of hard spots of learning is properly done and the suitable remedial measures are taken, the learning attainment as well as learning pace of the low achievers will certainly improve.

7.4.4 Summative Evaluation

Yet another type of evaluation is summative evaluation. It comes at the end of the course or the term. It involves a formal testing of the pupil's achievement and is used for grading, ranking, promoting and certifying the achievement of the students. It does not provide any scope for diagnosis and remediation.

7.5 Purpose of Evaluation

Evaluation has myriad of purpose in the field of education. Though, traditionally, evaluation used to be carried out for the purpose of determining mainly the achievement of students after the completion of teaching learning process during a fixed time interval, for pomoting to higher classes and for selection and certification the following may be listed as a few purposes for evaluation.

- To motivate students for better learning
- To diagnose strengths and weaknesses of students.
- To judge effectiveness of instruction.
- To adopt alternative approaches to improve learning.
- To help in determining as to how far the learning objectives could be achieved.
- To determine the rate of progress of students.
- To provide a base for certifying the students.
- To help in classifying the students.
- To predict the success of students in future.
- To help in selecting the students for admission in different subjects, different levels, scholarships, jobs etc.

7.6 Principles of Evaluation

Education, as it can be widely accepted, is a social and psychological process used in every field of lofe, day-by-day. An individual evaluating the behaviour or capability or potential of other individuals is also evaluating his/her own actions at regular intervals. Educational evaluation can be understood as measurement or assessment or both of all sound development of the pupil including his/her physcal, social, moral and intellectual development. Being a directional concept, educational evaluation has to follow certain principles.

Some of the prominent principles, which should be followed by educational evaluation can be:

1. *Continuity*

   The process of evaluation should be a continuous i.e., it should be a 24x7 process. And, all the activities must be seen in continuous approach in order to have better evaluation.

2. *Comprehensiveness*

   It should evaluate an individual or a group in a comprehensive manner. In simple words, it should consider physical, intellectual, emotional, social, moral, aesthetic and academic aspects of the evaluate.

3. *Totality*

Total behaviour of the learner, not only in parts, but in totality too, should be evaluated by the evaluation process. It should see the individual not only according to his/her current status but potential hidden their for future performance too.

4. *Spatial non-confinement*

As for as educational evaluation is concerned. It should be free from spatial boundaries. It should contain classroom activities as well as out door activities. Evaluation should not be confined to activities held in a classroom, but should also address the outdoor activities. Moreover evaluation can be anywhere irrespective of the place of instructions.

6. *Learning experiences*

Evaluation should be based on specific learning experiences of the learner. It should address the experiences, in context, of the learner. The context plays a vital role in teaching and learning. And, since, evaluation is an in built process of this process, the context must be taken care of by addressing previons and current experiences of the learner.

7. *Pupil- Centeredness*

The process of evaluation should be focussed on the pupil and his behaviour and cognition, not that of the evaluator. We are, still, struggling with the evaluation system which is centred towards the administration and needs of the pre-conceived, prejudiciously defined educational system. The pupil/learner is still not at the centre of the entire educational evaluation process. It will be an ideal situation if the evaluation would be by the the learner, for the learner and from the learner.

8. *Seclection of tools*

The tools must be selected in such a way that they are appropriate for the criteria to be assessed and in turn evaluated. There is no universal tool or panacea for all the levels, all the situations and all the systems. No single tool can address all the criteria of evaluation and hence tools must be selected keeping in mind the objectives, process and output of the evaluation.

9. *Objectivity*

The evaluation process must be objective. It should be free from biases. The objectivity becomes more challenging when there is a continuous and one-to-one evaluation. Moreover quatilative works evaluation needs a high degree of attention while objectivity is to be maintained in the evaluation process and the product.

The tools must be selected in such a way that they are appropriate for the criteria to be assessed and in turn evaluated.

7.7 Characteristics of a Good Evaluation Programme

From the above mentioned principles of evaluation, following characteristics of good evaluation can be derived :

1. *Evaluation should be an objective based process*

   The objective of evaluation is to measure educational achievement, which is reflected in terms of intended leaning outcomes or the instructional objectives. As such all evaluation must be geared to the instructional objectives, because objectives represent the product of learning. All instruments of evaluation must be based on instructional objectives for valid evaluation.

2. *Evaluation should be a continuous process*

   Since growth is a continuous process, the teacher/evaluator must remain cognizant of the changes that are taking place in the child's learning from time to time. Frequent or rather continued evaluation is, therefore, essential for getting reliable evidence about pupils' growth and development. Unless evaluation is made part and parcel of teaching-learning process, it cannot help in diagnosing pupils' difficulties and provide opportunities for remedial teaching and adoption of alternate strategies for teaching learning. Continuous evaluation augments improvement in learning. Therefore, it should not be merely at the end of the course activity.

3. *Evaluation should be comprehensive*

   The pupils have different dimensions of growth-intellectual, emotional and physical which are represented in the form of different objectives. Therefore, unless evaluation provides evidence on all the aspects, it cannot be considered comprehensive enough for decision making. Apart from testing all possible objectives, comprehensive evaluation involves use of different tools and techniques to get different types of evidences. Successful evaluation, therefore, need to be a comprehensive process.

4. *Evaluation should be an integral part of instruction*

   It is obvious that evaluation is part and parcel of instructional process. It should not be an end of the course activity rather it should be an inbuilt component of the whole process as it is integral in nature. Therefore, both instruction and evaluation must go hand in hand.

5. *Evaluation should be cooperative*

   Since comprehensive evaluation seeks evidence on all aspects of pupils' development, teacher alone cannot get full evidence about his growth. To collect evidence regarding social relationships, emotional behaviour, initiative, scientific attitudes, social attitudes, likes and dislikes, etc., the collaboration of pupils, his/her peers, parents and all those who watch him/her grow and develop is needed. Therefore for good evaluation, cooperation of different individuals and agencies is necessary.

6. *Evaluation should be a dynamic process*

   The dynamic process of evaluation signifies the changes in the level of objectives, instructions, and evaluation procedures. Evaluation is based on objectives of instruction, but at the same time it helps us to judge how far those objectives are attainable for a particular group of students. Once the objectives related to a subject specific content for a class during instructional span are attained, the new objectives are to be specified for further instruction. Simultaneously, new evaluation techniques have to be designed to meet the requirement of new objectives. This ensures dynamism in evaluation process.

7. *Evaluation should be a judgment-making process*

   At every step of the teaching-learning process appraisal is essential. Before the teaching-learning, it is necessary to determine the entry behaviour of students to decide for appropriate teaching-learning strategy to be adopted. During teaching-learning process, it is necessary to use the feedback of the results of the process on students' learning so as to ensure the success of further inputs. At the end of a unit or a course, it is, therefore, necessary to classify, grade or certify students on the basis of their performance. Thus, placement, formative, diagnostic and summative evaluation have to go on in tandem according to the purpose of evaluation.

If the schools keep in mind the above aspects of evaluation, there is no doubt that the evaluation system in our schools can be improved which in turn will help in improving the learning achievement of the students. The schools which carry out improved evaluation practices may really prove to be effective schools. The effective schools need to be strengthened to bring about improvement in the quality of education. It is certain that if evaluation is employed sincerely as a tool for quality improvement, undoubtedly, the excellence in performance of students would be achieved systematically. If there are any weaknesses in the learning, they can be plugged in time by providing interventions at the right time to improve the learning deficiency. Thus, evaluation will be a touchstone in the educational system.

UNIT 8

ASSESSING MATHEMATICS LEARNING

8.1 Introduction

Education reforms and subsequent recommendations of NCF–2005 have promoted changes in mathematics textbooks, teaching and testing while traditional mathematics education emphasised memorisation of mathematics facts and effective application of procedures. NCF–2005 calls for generating inner resources of learners to mathematise their experiences. There is a clear shift from remembering and recalling of mathematical facts to construct and understand mathematical concepts that demands deeper linkage of mathematical ideas in-terms of abstraction, structuration and generalisation. Problem solving as an essential component of mathematics education needs to be progressively woven during the different stages of school mathematics curriculum.

Sweeping changes in mathematics education is an outcome of changing perception about theories of learning, psychology of learners and our understanding of mathematics itself. It is now accepted that mathematics is product and process both; it is both an organised body of knowledge and a creative endeavour on the part of learner. In a series of progressive development, mathematics teachers and educators have reached a new consensus about the nature of real mathematics (Ernest, 1991). The key characteristics that distinguish mathematics from other domains of knowledge can be summerised as follows:

"Mathematics is the science and language of pattern....... As biology is a science of living organisms and physics is a science of matter and energy, so mathematics is a science of patterns...... To know mathematics is to investigate and express relationships among patterns, to describe patterns in complex and obscure contexts; to understand and transform relations among patterns : to classify, encode and describe patterns, to read and write in the language of patterns for various practical purposes" (Mathematical Science Educational Board, 1990, P.5).

The accepted view of mathematics as a basic arithmetic skills has given way to a broader view that emphasises mathematics as a general process, or way of thinking and reasoning (NCTM, 2000).

Current theories of learning mathematics suggest that students are not passive receivers of knowledge, but actively construct knowledge consensual with social and cultural setting (Van Glasersfield, 1991).

These changing peceptions have broadened the ways in which mathematics is taught as a discipline to promote logical thinking, investigation and problem solving. Learning mathematics extends beyond learning concepts, procedures and their applications. It includes developing right attitude and appreciation towards mathematics as a humanistic discipline. The learning of mathematics is a constructive process of meaningful experiences creating new spheres of mathematical ideas. It is also a cumulative process where new mathematical ideas are developed on already acquired mathematical ideas.

8.2 Assessment and Evaluation in Mathematics

Assessment is a critical component of complex, dynamic and continually adapting education process. In past several years, the concept of assessment is reconceptualised and new forms of assessment and evaluation have emerged. Instead of emphasising the end of course task mandate, the focus now is on making informed judgment about students' learning. Assessment requires systematic analysis of multiple types of evidences collected by the teacher to make judgments about students' learning, to diagnose learning difficulties and at the same time assessing effectiveness of her own teaching.

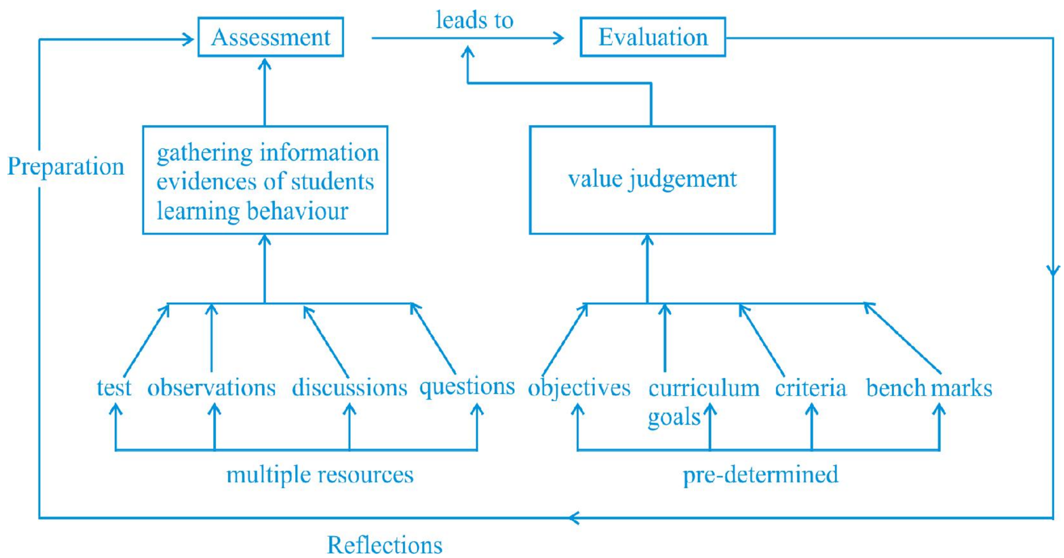
The changing view of assessment practices must be viewed as a consequence of our changing perception regarding mathematics teaching and mathematics learning. Mathematics itself is no longer seen as hierarchical and discrete (Stephens, 1992).

Assessment is a critical issue in the teaching and learning of mathematics that requires careful consideration to produce an assessment practice that reflects the learner's understanding of mathematical processes. This demands shift in the focus of assessment from summative assessment (where students are assessed to determine an overall measure of achievement) to the more supportive role of formative assessment (where students' learning behaviour in the classroom situation is meaningfully utilised in the pursuit of further learning).

Before we move further and deepen our understanding about various forms of assessment in mathematics, it is important to distinguish between the terms, 'assessment' and 'evaluation'. These terms are often used interchangeably creating confusion over their meanings and uses.

Assessment is a preleminary phase in the evaluation process. It is a systematic process to gather meaningful information and evidence about student's learning process through extensive means. Evaluation is the action phase in this process that allows teacher to weigh

assessment evidence with reference to some pre-set learning objectives/ criteria/ bench marks/curriculum goals. Evaluation may then lead to decisions, reflections and actions. So, the entire process of assessment and evaluation go hand-in-hand. Only authentic assessment leads to authentic evaluation, whereas assessment without evaluation serves no purpose as given below:



8.2.1 Assessment in Mathematics

" Assessment should support the learning of important mathematics and furnish useful information to both teachers and students," NCTM (1995).

The learning of mathematics, as discussed earlier, is a commulative process that promotes understanding mathematical knowledge. Meaningful mathematics learning reflects learners' ability to:

- use mathematics learning to communicate ideas
- integrate various aspects of mathematics knowledge
- connect concepts and processes
- mathematise their experiences
- abstract, generalise, reason, and analyse
- appreciate mathematics as a humanistic discipline.

A numerical score or grade assigned at limited frequency offers only a glimpse of students' knowledge. If the goal of assessment is to provide a valid and reliable profile of students' understanding and achievement, evidence must come from multiple sources, and at more frequent intervals.

Assessment should provide students opportunity to apply the mathematics they have learnt in real life contexts that go beyond the regular textbook. Students should view mathematics as a humanistic construct that permeates society and their own lives. This necessitates assessment activities aimed at promoting students' appreciation of mathematical world around them.

The valued sensitive educational reforms have placed ambitious targets to relook into the methods and strategies by which we assess students' progress. The higher aim of mathematics education that demands its deeper understanding by developing inner resources of learners clearly requires a change in mandate on evaluation system that reflects the scope and strength of learners.'

In the light of above discussion, evaluation in mathematics requires major paradigm shift as given below in the table No : 1

From	To
testing (for screening/labelling)	assessment for informed decission making
few discrete	seamless integration of instructional practices and assessment
behavioural objectives assessment	constructive objectives
one numerical score	multi-dimensional profiles
right answer approach	reasoned answer approach
grading and report cards	cumulative learning progress graph.

Assessment must be an integral part of the social constructivist theory of learning that is supported by various reform documents, and mathematics education research all around the world.

We need those assessment practices that support students' construction of knowledge and respect the diversity among learners. The assessment objectives should shift from routine mathematical facts and skills to conceptual understanding, procedural knowledge, appreciation and application, personal beliefs and attitude towards mathematics. The learners are the rich resource of cultural experiences which they bring to the class. "The recognition of such diversity has necessitated a shift in the vision of evaluation towards a system based on evidence from multiple sources and away from using externally derived evidence" (NCTM, 1995).

8.3 Continuous and Comprehensive Assessment (CCA)

CCA refers to continuous and comprehensive Assessment scheme that covers diverse attributes of students' learning growth. Continuous means Assessment as an ongoing process which assesses learner's performance on a continuum.

Comprehensive means wide spectrum of assessment opportunities to evaluate learner's diverse interests and abilities, covering all aspects of learning.

Assessment as an interwoven process of teaching, allows teacher to do *meaningful evaluation* about students' current learning and possible future learning. The current education reforms strongly advocate continuous and comprehensive Assessment system making evaluation a more fluid and dynamic approach.

Components of CCA

(A) Formative Assessment

(B) Summative Assessment

Value judgment

8.3.1 Formative Assessment

Formative Assessment is an ongoing classroom process that keeps students and teachers informed of students' progress towards achieving the learning objectives. Formative assessments are every day classroom opportunities that teachers employ develop during classroom discourses to :

- gain an insight into students' present level of mathematical understanding.
- identify the gap between students' current level of understanding and the derived aims
- modify ongoing instructional plan to bridge the gap in accordance to students' learning needs.

These assessments are called formative, because they exhibit the formation of learning that emerges through daily instructional activity.

According to research synthesis by Black and Williams (year ?), "Only when assessment reveals specifics about students' thinking in ways that inform future learning experiences is the assessment formative."

The main purpose of formative assessment is to provide insights into students' thinking, enabling teachers to monitor changes in students' thinking and reasoning during classroom discourses. It provides teachers with valuable information upon which instructional modifications can be made. Effective formative assessment enhances students' learning in several ways. It enables students to value meaningful and worthy mathematical knowledge.

When teachers use multiple modes of assessment strategies such as observations, interviews and interactions, students become active participants in the process of evaluation, assuming responsibility for their own learning and becoming more confident and independent learners. When teachers gather and reflect on useful information about what students are learning, they can always shape their instructions towards achieving significant mathematical goals. Formative assessment is an asset for the teacher as it empowers teachers to take important decisions such as when to revisit a difficult concept, when to review instructions for students who are struggling or how to extend learning for those who need enrichment. Formative assessment is the most fluid and primary source of evidence that guides a teacher to make their inferences and take decisions. To add value to the information gathered during formative assessment, teacher should move beyond narrow analysis of responses in terms of right or wrong to a more focussed and descriptive analysis of responses to identify students' insight on which further objectives can be planned. Authentic evaluation of students' understanding require methods used on everyday classroom basis as well as those used on a less frequent basis. Using a variety of assessment strategies in a phased manner reflects our sensitivity to handle cognitive and cultural diversity among students.

Clearly defined learning goals and measurable instructional objectives are the basis for planning effective formative evaluation. Many of the evidence gathered during formative evaluation are of descriptive in nature that can be matched with instructional targets.

Methods of Formative Assessment

As already stated, formative assessment involves direct interaction of teacher with students during teaching discourses but also an extension of it outside the class. The process requires observation, listening to student's responses, peer dialogue and making effective judgment. It will also include traditional written tests, checklists and rubric to assess students' performance. There are several assessment opportunities which teachers must utilise to understand mathematical learning. Assessment opportunities during classroom discourses include :

- when a student interacts with another student
- when students ask questions about the content being taught in the class
- when students answer to the questions posed by the teacher
- when students apply the learning to another situation
- when students formulate their own argument during classroom discourses
- when students feel excited about learning and enjoy the class.

The above situations are examples of formative assessment methods of monitoring learning progress and shaping instructional practices accordingly.

Let us now discuss some of these methods:

(A) Observing Students

The students' non verbal behaviour during classroom discourses (froweners, puzzled looks, shaking head, lit up eyes, raising hand, looking elsewhere, passive mood, avoiding eye contact) are the clear indicators to know whether student has understood the concept or not. Observation of learning behaviour is an effective tool to determine the learner's level of involvement and interest in the learning activity. It can be effectively done by involving students in small group activities in cooperative learning set up and keenly observe their participation and involvement to gain insight. One useful device is to prepare an observation schedule that teacher can use to record students' behaviour during cooperative learning activity. Teacher spends few minutes observing each group and individual student within each group and then record instances that exhibit students' understanding clarity, confidence, creativity, intuition and motivation regarding the mathematics task they are involved in the table below:

Suggestive Observation Schedule
(using rating scale)

Name :

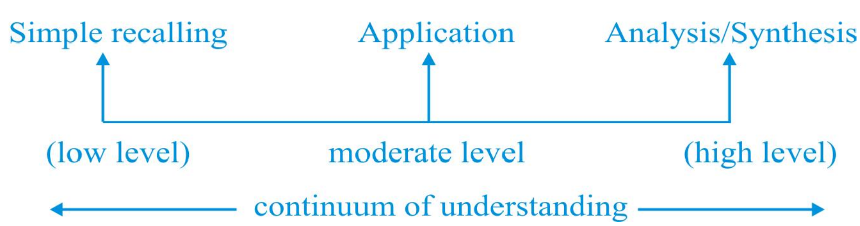
Class :

Task :

Learning behaviours	Always	Frequent	Sometimes	Rarely	Never
Active listening					
Staying on task					
Initiating discussion					
Asking questions					
Guessing questions					
Answering					
Justifying the answers					

(B) Questioning

The instructional situation can be effectively used as an informal assessment strategy to gauge students' level of understanding. The questioning can range from lower level of understanding to higher level of understanding.



The sequencing of questions can be done on the continuum of understanding of what learners know about mathematics, what they think about mathematics and how they can use mathemaitcs. These systematically phased questions become rich resource to use student's responses for further dialogue, peer collaboration and other kinds of extensions. Incorrect answers may be rectified by other students and teachers may ask probing questions to diagnose the reason for incorrect response. Correct response may allow teachers to move to next level of question or extending the same question.

Points to remember for effective questioning:

- using wide range of questions
- using wide level of questions
- using appropriate wait time
- redirecting questions
- creating secure learning environment.

It should not turn out a situation where few students dominate the class and others feel left out or shy away as they are not sure of their answers. Teachers should direct the question session in a highly democratic and supportive manner where every student feel important and every response adds value to the session. Objective of questioning is not to get the right answer but enabling students to reach to the all possible answers which are mathematically correct.

Example

Learning Context : Coprime numbers

Q1 What are coprime numbers? (Recalling)

Q2 Is a consecutive pair of numbers coprime?

Q3 Do coprime numbers always occur in pair? (Extension I)

Q4 Do coprime numbers form a consecutive pair? (Extension II)

Q5 Using Euclid's division algorithm, find which of the following pair of numbers are coprime:

(i) 231, 396 (ii) 847, 2160 (Application)

Q6 What is the highest common factor of a coprime pair? (Analysis)

Q7 Can you suggest any other method to find coprime pairs? (Creative)

(C) Students' Questions

'Only an aroused mind can raise questions'

In a stimulating learner centred classroom, teachers are asked to encourage students to raise their doubts and questoins. Students' questions are the opening points that allow teachers to gauge students' doubts and understanding at the same time. Listening to the kinds and levels of questions raised by students and initiating discussion in the class based on those questions to elicit response from other students can provide valuable information to the teacher to make formative assessment.

Example:

Learning context : Quadratic Equations

Definition: A quadratic equation in the variable x is of the form $ax^2 + bx + c = 0$, where a, b, c are real numbers and $a \neq 0$.

Probable questions raised by students

Q1 Can we write a quadratic equation with more than one variable?

Q2 What if 'a' assumes negative value?

Q3 What would be the nature of quadratic equation if 'b' is zero (0)?

Such questions can provide active platform to initiate discussion and exploration in the class. Redirecting questions for peer dialogue is an effective strategy for formative assessment.

(D) Individual Small group interaction

It is a well researched fact that many students feel shy in the class and don't participate even after frequent encouragment from the teachers. Meeting with students individually or in small groups is very effective as students feel themselves important and come out more openly in such interactions. Discussion is an effective means of judging a students' ability as a productive participant in small group. Teachers can use this opportunity to assess student's specific habits of mathematics learning which can be further mentored. Such interactions allow teachers to dignose difficulties of students during course of mathematics learning. Such interactions are important to value diversity in the class.

These provide opportunities to explain students specific learning behaviour, personal beliefs, attitude and interests. It can be used to allow students to analyse the classroom processes.

The possible questions a teacher may ask:

Q1 What do you think about your mathematics class?

Q2 Which are the recent topics you have enjoyed and feel confident?

Q3 Which are the topics you find difficult?

Q4 Mention your

- favourite mathematics problem
- difficult mathematics problem
- strength in mathematics
- weakness in mathematics.

Q5 What do you enjoy the most in mathematics class?

- mental mathematics
- problem solving
- theorems
- projects/activities/experiments
- group work.

(E) Content Specific Assignments/Worksheets

Formative assessment is effective only when teachers know their students more closely in terms of their learning behaviours, learning capacity, interests and personal beliefs. Individual work sheets can be planned based on the learning needs of each student. Such work sheets can be structured around the choice of complex, multifaceted tasks to allow students to answer at different levels of sophistication. If students are to perform at their maximum level of ability, the measures by which they are judged should give them such opportunities. Prior knowledge, experience and the opportunities to the learner are important considerations in interpreting assessment results.

8.3.2 Summative Assessment

Summative Assessment occurs at the end of a term or course of study. Its main purpose is to evaluate the amount of learning over a period of time, to summarise students' progress, and to make judgment of the learning progress relative to curriculum objectives. It is difficult to draw a clear demarcation between strictly formative or strictly summative assessments. It depends upon the objectives of evaluation as summative assessment can also be used formatively to help teachers in making decisions.

Theortically, summative assessment means that students would be evaluated using a

'test.' Currently, summative assessment has been undergoing major reforms. These reforms include the use of variety of methods including performance based assessment, test papers, activity based questions, projects and students portfolios.

Performance Based Assessment

Performance based assessment typically involves students performing either individually or in small groups in the task of problem solving, in discussion forum about a mathematical context or working towards a mathematical understanding. It also involves assessing student's:

- understanding of mathematical concepts
- ability to communicate mathematically
- ability to discriminate between relevant and irrelevant attribute of a concept
- ability to represent concepts in variety of ways
- ability to apply concepts in multiple contexts
- ability to use mathematics vocabulary with clarity, precision and appropriateness.

Performance based assessment is a creative approach that allow students to 'do mathematics' in applied situation, to 'think about mathematics' and to 'reflect on mathematical structures and procedures'. Following are some chracterstics of performance based assessment :

- provide students with real world experiences
- involve students in sustained mathematical task, spread over few days
- focus on the coherent and holistic understanding of mathematics, rather than isolated concepts of mathematics
- generally theme based task involving several mathematical concepts and procedures
- requires precision, clarity and effective use of mathematics vocabulary
- stimulate students for innovations, experimentation and reflections
- strengthening ability to make connections among important aspects of mathematics learning.

Performance based assessment can be scored on well defined rubric using set of criteria related to content, process, organisation, creativity and communication.

Task: Draw ten circles of differenrt radii and centres. Draw pair of intersecting chords in each circle bisecting each other. Comment on the nature of chords so formed. This task can be assessed using the following rubric in which various performance criteria are used.

Criteria	4	3	2	1
Content comprehension	Figures are drawn with high level of accuracy and perfect labelling	Most of the figures are drawn with high level of accuracy and perfect labelling	Most of the figures are drawn with accuracy and labelling	Some Figures are drawn with accuracy and labelling
Creativity	Figures are drawn in the most varied perspectives, measurement and intersection	Most of the figures are drawn with varied perspectives	Some of the figures are drawn with varied perspectives	Figures are drawn in a routine manner
Organisation	All relevant information is organised systematically to reach to conclusion	Most of the information is organised systematically	Some of the information is organised systematically	Information is unorganised and appears haphazard
Process	All relevant information is used to analysis, generalise and synthesis in the results	Most of the information is used for the process of analysis, generalisation and synthesis	Only partial information is used for the process of analysis, generalisation and synthesis	Serious flaws in the process of analysis, generalisation and synthesis
Communication	Valid inferences are made using precise and appropriate mathematics vocabulary	Valid inferences are made using mathematics vocabulary	Reasonable inferences are made using mainly mathematics vocabulary	Logically inappropriate or incomplete inferences are made and arbitrary use of mathematics vocabulary

Assessing meaningful mathematics learning is a creative endeavour. We have explored several aspects of assessment in this unit that empowers mathematics teacher to make valid judgments about students' learning and effectiveness of teaching.

8.4 Learning Exercises for Teachers

(1) What are the important components of a comprehensive and continuous assessment strategy?

(2) Select a unit from Class IX/X secondary mathematics text and create a formative assessment plan.

(3) How does performance based assessment differ from traditional methods of summative assessment.

(4) Plan a situation where summative assessment can be used as formative assessment.

(5) Plan an effective rubric for performance based assessment. Use a suitable example as illustration.

(6) Plan a meaningful small group activity from number system/statistics. Prepare an effective observation schedule using rating scale that can be used by the teacher during the activity.

(7) Plan a series of questions at different levels of continuum of understanding for a suitable topic of your choice.