(6 Mark Questions)

EXERCISE 1.1

(2) Find the adjoint of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}$ and verify the result $A(\text{adj}\,A) = (\text{adj}\,A)\,A = |A|\,.\,I$

(4) Find the inverse of each of the following matrices :

(i) $\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

(iv) $\begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$ (v) $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

(5) If $A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ verify that(i) $(AB)^{-1} = B^{-1}A^{-1}$ (ii) $(AB)^T = B^T A^T$

(8) Show that the adjoint of $A = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$ is A itself.

(10) For $A = \begin{bmatrix} -1 & 2 & -2 \\ 4 & -3 & 4 \\ 4 & -4 & 5 \end{bmatrix}$ show that $A = A^{-1}$.

(1) State and prove reversal law for inverses of matrices.

EXERCISE 1.2.

Solve by matrix inversion method each of the following system of linear equations:

(1) $2x - y = 7,\ 3x - 2y = 11$ (2) $7x + 3y = -1,\ 2x + y = 0$

EXERCISE 1.3.

Find the rank of the following matrices:

(1) $\begin{bmatrix} 1 & 1 & -1 \\ 3 & -2 & 3 \\ 2 & -3 & 4 \end{bmatrix}$ (2) $\begin{bmatrix} 6 & 12 & 6 \\ 1 & 2 & 1 \\ 4 & 8 & 4 \end{bmatrix}$ (3) $\begin{bmatrix} 3 & 1 & 2 & 0 \\ 1 & 0 & -1 & 0 \\ 2 & 1 & 3 & 0 \end{bmatrix}$

(4) $\begin{bmatrix} 0 & 1 & 2 & 1 \\ 2 & -3 & 0 & -1 \\ 1 & 1 & -1 & 0 \end{bmatrix}$ (5) $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{bmatrix}$ (6) $\begin{bmatrix} 1 & -2 & 3 & 4 \\ -2 & 4 & -1 & -3 \\ -1 & 2 & 7 & 6 \end{bmatrix}$

EXERCISE 1.4.

(1) Solve the following non-homogeneous system of linear equations by determinant method:

(iii) $4x + 5y = 9$; $8x + 10y = 18$

EXERCISE 1.5.

(1) Examine the consistency of the following system of equations. If it is consistent then solve the same.

(iii) $x+y+z=7$; $x+2y+3z=18$; $y+2z=6$

(iv) $x-4y+7z=14$; $3x+8y-2z=13$; $7x-8y+26z=5$

Example 1.2:

Find the adjoint of the matrix $A=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}$

Example 1.3:

If $A=\begin{bmatrix}-1 & 2\\ 1 & -4\end{bmatrix}$, verify the result $A(adj\ A)=(adj\ A)A=|A|I_2$

Example 1.5:

Find the inverse of the following matrix:

(iv) $A=\begin{bmatrix}3 & 1 & -1\\ 2 & -2 & 0\\ 1 & 2 & -1\end{bmatrix}$

Example 1.6:

If $A=\begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}$ and $B=\begin{bmatrix}0 & -1\\ 1 & 2\end{bmatrix}$ verify that $(AB)^{-1}=B^{-1}A^{-1}$.

Example 1.7:

Solve by matrix inversion method $x+y=3$, $2x+3y=8$

Example 1.12:

Find the rank of the matrix $\begin{bmatrix}1 & 1 & 1 & 3\\ 2 & -1 & 3 & 4\\ 5 & -1 & 7 & 11\end{bmatrix}$

Example 1.13:

Find the rank of the matrix $\begin{bmatrix}1 & 1 & -1\\ 2 & -3 & 4\\ 3 & -2 & 3\end{bmatrix}$

Example 1.14:

Find the rank of the matrix $\begin{bmatrix}1 & 2 & 3 & -1\\ 2 & 4 & 6 & -2\\ 3 & 6 & 9 & -3\end{bmatrix}$

Example 1.15:

Find the rank of the matrix $\begin{bmatrix}4 & 2 & 1 & 3\\ 6 & 3 & 4 & 7\\ 2 & 1 & 0 & 1\end{bmatrix}$

Example 1.16:

Find the rank of the matrix $\begin{bmatrix}3 & 1 & -5 & -1\\ 1 & -2 & 1 & -5\\ 1 & 5 & -7 & 2\end{bmatrix}$

Example 1.17:

Solve the following system of linear equations by determinant method.

(2) $2x+3y=8$; $4x+6y=16$

Example 1.18:

Solve the following non-homogeneous equations of three unknowns.

(3) $2x+2y+z=5$; $x-y+z=1$; $3x+y+2z=4$

(5) $x+y+2z=4$; $2x+2y+4z=8$; $3x+3y+6z=10$

EXERCISE 2.2

Prove by vector method

(1) If the diagonals of a parallelogram are equal then it is a rectangle.

(2) The mid point of the hypotenuse of a right angled triangle is equidistant from its vertices.

(3) The sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of the sides.

(8) Forces of magnitudes 3 and 4 units acting in the directions $6\vec{i}+2\vec{j}+3\vec{k}$ and $3\vec{i}-2\vec{j}+6\vec{k}$ respectively act on a particle which is displaced from the point (2,2-1) to (4,3,1). Find the work done by the forces.

EXERCISE 2.3

(9) Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be unit vectors such that $\vec{a}.\vec{b}=\vec{a}.\vec{c}=0$ and the angle between $\vec{b}$ and $\vec{c}$ is $\frac{\pi}{6}$. Prove that $\vec{a}=\pm 2\left(\vec{b}\times\vec{c}\right)$

EXERCISE 2.4.

(5) Prove by vector method that the parallelogram on the same base and between the same parallels are equal in area.

(6) Prove that twice the area of a parallelogram is equal to the area of another parallelogram formed by taking as its adjacent sides the diagonals of the former parallelogram.

(8) Forces $2\vec{i}+7\vec{j}$, $2\vec{i}-5\vec{j}+6\vec{k}$, $-\vec{i}+2\vec{j}-\vec{k}$ act at a point P whose position vector is $4\vec{i}-3\vec{j}-2\vec{k}$. Find the moment of the resultant of three forces acting at P about the point Q whose position vector is $6\vec{i}+\vec{j}-3\vec{k}$.

(10) Find the magnitude and direction cosines of the moment about the point (1,-2,3) of a force $2\vec{i}+3\vec{j}+6\vec{k}$ whose line of action passes through the origin.

EXERCISE 2.5.

(4) Show that the points (1,3,1) , (1,1,-1) , (-1,1,1) (2,2,-1) are lying on the same plane. (Hint : It is enough to prove any three vectors formed by these four points are coplanar).

(7) If $\vec{a}=2\vec{i}+3\vec{j}-5\vec{k}$, $\vec{b}=-\vec{i}+\vec{j}+2\vec{k}$ and $\vec{c}=4\vec{i}-2\vec{j}+3\vec{k}$, show that $\left(\vec{a}\times\vec{b}\right)\times\vec{c}\neq\vec{a}\times\left(\vec{b}\times\vec{c}\right)$

(8) Prove that $\left(\vec{a}\times\vec{b}\right)\times\vec{c}=\vec{a}\times\left(\vec{b}\times\vec{c}\right)$ iff $\vec{a}$ and $\vec{c}$ are collinear.(vector triple products is non-zero)

(10) Prove that $\left(\vec{a}\times\vec{b}\right).\left(\vec{c}\times\vec{d}\right)+\left(\vec{b}\times\vec{c}\right).\left(\vec{a}\times\vec{d}\right)+\left(\vec{c}\times\vec{a}\right).\left(\vec{b}\times\vec{d}\right)=0$

(11) Find $\left(\vec{a}\times\vec{b}\right).\left(\vec{c}\times\vec{d}\right)$ if $\vec{a}=\vec{i}+\vec{j}+\vec{k}$, $\vec{b}=2\vec{i}+\vec{k}$, $\vec{c}=2\vec{i}+\vec{j}+\vec{k}$, $\vec{d}=\vec{i}+\vec{j}+2\vec{k}$

EXERCISE 2.6.

(4) A vector $\vec{r}$ has length $35\sqrt{2}$ and direction ratios (3,4,5) , find the direction cosines and components of $\vec{r}$.

(6) Find the vector and Cartesian equation of the line through the point (3,-4,2) and parallel to the vector $9\vec{i}+6\vec{j}+2\vec{k}$.

(7) Find the vector and Cartesian equation of the line joining the points (1,-2,1) and (0,-2,3)

EXERCISE 2.7.

(1) Find the shortest distance between the parallel lines

(i) $\bar{r} = \left(2\vec{i} - \vec{j} - \vec{k}\right) + t\left(\vec{i} - 2\vec{j} + 3\vec{k}\right)$ and $\bar{r} = \left(\vec{i} - 2\vec{j} + \vec{k}\right) + s\left(\vec{i} - 2\vec{j} + 3\vec{k}\right)$

(ii) $\frac{x-1}{-1} = \frac{y}{3} = \frac{z+3}{2}$ and $\frac{x-3}{-1} = \frac{y+1}{3} = \frac{z-1}{2}$

(2) Show that the following two lines are skew lines:
$\bar{r} = \left(3\vec{i} + 5\vec{j} + 7\vec{k}\right) + t\left(\vec{i} - 2\vec{j} + \vec{k}\right)$ and $\bar{r} = \left(\vec{i} + \vec{j} + \vec{k}\right) + s\left(7\vec{i} + 6\vec{j} + 7\vec{k}\right)$

(4) Find the shortest distance between the skew lines $\frac{x-6}{3} = \frac{y-7}{-1} = \frac{z-4}{1}$ and $\frac{x}{-3} = \frac{y+9}{2} = \frac{z-2}{4}$

(5) Show that (2,-1,3) , (1,-1,0) and (3,-1,6) are collinear.

EXERCISE 2.8.

(1) Find the vector and Cartesian equations of a plane which is at a distance of 18 units from the origin and which is normal to the vector $2\vec{i} + 7\vec{j} + 8\vec{k}$

(4) The foot of the perpendicular drawn from the origin to a plane is (8,-4,3). Find the equation of the plane.

(5) Find the equation of the plane through the point whose p.v. is $2\vec{i} - \vec{j} + \vec{k}$ and perpendicular to the vector $4\vec{i} + 2\vec{j} - 3\vec{k}$.

(6) Find the vector and Cartesian equations of the plane through the point (2,-1,4) and parallel to the plane $\bar{r} \cdot \left(4\vec{i} - 12\vec{j} - 3\vec{k}\right) = 7$.

(15) Find the Cartesian form of the following planes:

(i) $\bar{r} = (s-2t)\vec{i} + (3-t)\vec{j} + (2s+t)\vec{k}$

(ii) $\bar{r} = (1+s+t)\vec{i} + (2-s+t)\vec{j} + (3-2s+2t)\vec{k}$

EXERCISE 2.9.

(1) Find the equation of the plane which contains the two lines $\frac{x+1}{2} = \frac{y-2}{-3} = \frac{z-3}{4}$ and $\frac{x-4}{3} = \frac{y-1}{2} = z-8$

(2) Can you draw a plane through the given two lines? Justify your answer.
$\bar{r} = \left(\vec{i} + 2\vec{j} - 4\vec{k}\right) + t\left(2\vec{i} + 3\vec{j} + 6\vec{k}\right)$ and $\bar{r} = \left(3\vec{i} + 3\vec{j} - 5\vec{k}\right) + s\left(-2\vec{i} + 3\vec{j} + 8\vec{k}\right)$

(3) Find the point of intersection of the line
$\bar{r} = \left(\vec{j} - \vec{k}\right) + s\left(2\vec{i} - \vec{j} + \vec{k}\right)$ and xz – plane.

(4) Find the meeting point of the line $\bar{r} = \left(2\vec{i} + \vec{j} - 3\vec{k}\right) + t\left(2\vec{i} - \vec{j} - \vec{k}\right)$ and the plane $x - 2y + 3z + 7 = 0$

EXERCISE 2.11

(1) Find the vector equation of a sphere with centre having position vector $2\vec{i} - \vec{j} + 3\vec{k}$ and radius 4 units. Also find the equation in Cartesian form.

(2) Find the vector and Cartesian equation of the sphere on the join of the points A and B having position vectors $2\vec{i} + 6\vec{j} - 7\vec{k}$ and $-2\vec{i} + 4\vec{j} - 3\vec{k}$ respectively as a diameter. Find also the centre and radius of the sphere.

(3) Obtain the vector and Cartesian equation of the sphere whose centre is (1, -1, 1) and radius is the same as that of the sphere $\left|\bar{r} - \left(\vec{i} + \vec{j} + 2\vec{k}\right)\right| = 5$.

(6) Show that diameter of a sphere subtends a right angle at a point on the surface by vector method.

Example 2.12:

With usual notations in triangle ABC prove that $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

Example 2.13:

With usual notations prove (i) $a = b\cos C + c\cos B$

Example 2.14:

Angle in a semi-circle is a right angle. Prove by vector method.

Example 2.15:

Diagonals of a rhombus are at right angles. Prove by vector methods.

Example 2.24:

If $\vec{p} = -3\vec{i} + 4\vec{j} - 7\vec{k}$ and $\vec{q} = 6\vec{i} + 2\vec{j} - 3\vec{k}$ then find $\vec{p} \times \vec{q}$. Verify that $\vec{p}$ and $\vec{p} \times \vec{q}$ are perpendicular to each other and also verify that $\vec{q}$ and $\vec{p} \times \vec{q}$ are perpendicular to each other.

Example 2.25:

If the position vectors of three points A, B and C are respectively $\vec{i} + 2\vec{j} + 3\vec{k}$, $4\vec{i} + \vec{j} + 5\vec{k}$ and $7\left(\vec{i} + \vec{k}\right)$. Find $\overrightarrow{AB} \times \overrightarrow{AC}$. Interpret the result geometrically.

Example 2.26:

Prove that the area of a quadrilateral $\frac{1}{2}\left|\overrightarrow{AC} \times \overrightarrow{BD}\right|$ where AC and BD are its diagonals.

Example 2.27:

If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of the vertices A, B, C of a triangle ABC, then prove that the area of triangle ABC is $\frac{1}{2}\left|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}\right|$ Deduce the condition for the points $\vec{a}, \vec{b}, \vec{c}$ to be collinear.

Example 2.28:

Prove that in triangle ABC with usual notations, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ by vector method.

Example 2.33:

For any three vectors $\vec{a}, \vec{b}, \vec{c}$ prove that $\left[\vec{a} + \vec{b},\ \vec{b} + \vec{c},\ \vec{c} + \vec{a}\right] = 2\left[\vec{a}\ \vec{b}\ \vec{c}\right]$

Example 2.36:

If $\vec{a} = 3\vec{i} + 2\vec{j} - 4\vec{k}$, $\vec{b} = 5\vec{i} - 3\vec{j} + 6\vec{k}$, $\vec{c} = 5\vec{i} - \vec{j} + 2\vec{k}$,find (i) $\vec{a} \times \left(\vec{b} \times \vec{c}\right)$ (ii) $\left(\vec{a} \times \vec{b}\right) \times \vec{c}$ and show that they are not equal.

Example 2.37:

Let $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ be any four vectors then

(i) $\left(\vec{a} \times \vec{b}\right) \times \left(\vec{c} \times \vec{d}\right) = \left[\vec{a}\ \vec{b}\ \vec{d}\right]\vec{c} - \left[\vec{a}\ \vec{b}\ \vec{c}\right]\vec{d}$

(ii) $\left(\vec{a} \times \vec{b}\right) \times \left(\vec{c} \times \vec{d}\right) = \left[\vec{a}\ \vec{c}\ \vec{d}\right]\vec{b} - \left[\vec{b}\ \vec{c}\ \vec{d}\right]\vec{a}$

Example 2.38:

Prove that $\left[\vec{a} \times \vec{b},\ \vec{b} \times \vec{c},\ \vec{c} \times \vec{a}\right] = \left[\vec{a},\ \vec{b},\ \vec{c}\right]^2$

Example 2.39:

Find the vector and Cartesian equations of the straight line passing through the point A with position vector $3\vec{i} - \vec{j} + 4\vec{k}$ and parallel to the vector $-5\vec{i} + 7\vec{j} + 3\vec{k}$

Example 2.40:

Find the vector and Cartesian equations of the straight line passing through the points (-5,2,3) and (4,-3,6)

Example 2.42:

Find the shortest distance between the parallel lines

$\bar{r} = \left(\vec{i} - \vec{j}\right) + t\left(2\vec{i} - \vec{j} + \vec{k}\right)$ and $\bar{r} = \left(2\vec{i} + \vec{j} + \vec{k}\right) + s\left(2\vec{i} - \vec{j} + \vec{k}\right)$

Example 2.43:

Show that the two lines $\overline{r} = \left(\vec{i} - \vec{j}\right) + t\left(2\vec{i} + \vec{k}\right)$ and $\overline{r} = \left(2\vec{i} - \vec{j}\right) + s\left(\vec{i} + \vec{j} - \vec{k}\right)$ are skew lines and find the distance between them.

Example 2.45:

Find the shortest distance between the skew lines
$\overline{r} = \left(\vec{i} - \vec{j}\right) + \lambda\left(2\vec{i} + \vec{j} + \vec{k}\right)$ and $\overline{r} = \left(\vec{i} + \vec{j} - \vec{k}\right) + \mu\left(2\vec{i} - \vec{j} - \vec{k}\right)$

Example 2.46:

Show that the points (3,-1,-1) , (1,0,-1) and (5,-2,-1) are collinear.

Example 2,48:

Find the vector and Cartesian equation of a plane which is at a distance of 8 units from the origin and which is normal to the vector $3\vec{i} + 2\vec{j} - 2\vec{k}$

Example 2.49:

The foot of perpendicular drawn from the origin to the plane is (4,-2,-5) , find the equation of the plane.

Example 2.53:

Find the equation of the plane passing through the line of intersection of the plane $2x - 3y + 4z = 1$ and $x - y = -4$ and passing through the point (1,1,1)

Example 2.54:

Find the equation of the plane passing through the intersection of the planes $2x - 8y + 4z = 3$ and $3x - 5y + 4z + 10 = 0$ and perpendicular to the plane $3x - y - 2z - 4 = 0$

Example 2.55:

Find the distance from the point (1,-1,2) to the plane $\overline{r} = \left(\vec{i} + \vec{j} + \vec{k}\right) + s\left(\vec{i} - \vec{j}\right) + t\left(\vec{j} - \vec{k}\right)$

Example 2.56:

Find the distance between the parallel planes $\overline{r} \cdot \left(-\vec{i} - \vec{j} + \vec{k}\right) = 3$ and $\overline{r} \cdot \left(\vec{i} + \vec{j} - \vec{k}\right) = 5$

Example 2.57:

Find the equation of the plane which contains the two lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = \frac{z}{1}$

Example 2.58:

Find the point of intersection of the line passing through the two points (1,1,-1) ; (-1,0,1) and the xy -plane.

Example 2.59:

Find the co-ordinates of the point where the line $\overline{r} = \left(\vec{i} + 2\vec{j} - 5\vec{k}\right) + t\left(2\vec{i} - 3\vec{j} + 4\vec{k}\right)$ meets the plane
$\overline{r} \cdot \left(2\vec{i} + 4\vec{j} - \vec{k}\right) = 3$

Example 2.62:

Find the vector and Cartesian equations of the sphere whose centre is $2\vec{i} - \vec{j} + 2\vec{k}$ and radius is 3.

Example 2.63:

Find the vector and Cartesian equation of the sphere whose centre is (1,2,3) and which passes through he point (5,5,3)

Example 2.64:

Find the equation of the sphere on the join of the points A and B having position vectors $2\vec{i} + 6\vec{j} - 7\vec{k}$ and $2\vec{i} - 4\vec{j} + 3\vec{k}$ respectively as a diameter.

Example 2.65:

Find the coordinates of the centre and the radius of the sphere whose vector equation is
$\overline{r}^2 - \overline{r} \cdot \left(8\vec{i} - 6\vec{j} + 10\vec{k}\right) - 50 = 0$

EXERCISE 3.1.

(1) Express the following in the standard form $a+ib$

(ii) $\dfrac{(1+i)(1-2i)}{1+3i}$ (iv) $\dfrac{i^4+i^9+i^{16}}{3-2i^8-i^{10}-i^{15}}$

(4) Find the real values of x and y for which the following equations are satisfied.

(i) $(1-i)x+(1+i)y=1-3i$ (ii) $\dfrac{(1+i)x-2i}{3+i}+\dfrac{(2-3i)y+i}{3-i}=i$ (iii) $\sqrt{x^2+3x+8}+(x+4)i=y(2+i)$

(5) For what values of x and y, the numbers $-3+ix^2y$ and x^2+y+4i are complex conjugate of each other?

EXERCISE 3.2.

(2) Find the square root of $(-8-6i)$

(4) Prove that the triangle formed by the points representing the complex numbers $(10+8i)$, $(-2+4i)$ and $(-11+31i)$ on the Argand plane is right angled.

(5) Prove that the points representing the complex numbers $(7+5i)$, $(5+2i)$, $(4+7i)$ and $(2+4i)$ form a parallelogram.

(Plot the points and use midpoint formula).

(7) If $\arg(z-1)=\dfrac{\pi}{6}$ and $\arg(z+1)=2\dfrac{\pi}{3}$ then prove that $|z|=1$

(8) P represents the variable complex number z. Find the locus of P, if (ii) $|z-5i|=|z+5i|$ (iv) $|2z-3|=2$.

EXERCISE 3.3.

(1) Solve the equation $x^4-8x^3+24x^2-32x+20=0$ if $3+i$ is a root.

(2) Solve the equation $x^4-4x^3+11x^2-14x+10=0$ if one root is $1+2i$

(3) Solve : $6x^4-25x^3+32x^2+3x-10=0$ given that one of the root is $2-i$

EXERCISE 3.4.

(2) Simplify: $\dfrac{(\cos\alpha+i\sin\alpha)^3}{(\sin\beta+i\cos\beta)^4}$

(3) If $\cos\alpha+\cos\beta+\cos\gamma=0=\sin\alpha+\sin\beta+\sin\gamma$,prove that

(i) $\cos 3\alpha+\cos 3\beta+\cos 3\gamma=3\cos(\alpha+\beta+\gamma)$

(ii) $\sin 3\alpha+\sin 3\beta+\sin 3\gamma=3\sin(\alpha+\beta+\gamma)$

(iii) $\cos 2\alpha+\cos 2\beta+\cos 2\gamma=0$

(iv) $\sin 2\alpha+\sin 2\beta+\sin 2\gamma=0$

(v) $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=\sin^2\alpha+\sin^2\beta+\sin^2\gamma=\dfrac{3}{2}$

(4) Prove that

(i) $(1+i)^n+(1-i)^n=2^{\frac{n+2}{2}}\cos\dfrac{n\pi}{4}$

(ii) $(1+i\sqrt{3})^n+(1-i\sqrt{3})^n=2^{n+1}\cos\dfrac{n\pi}{3}$

(iii) $(1+\cos\theta+i\sin\theta)^n+(1+\cos\theta-i\sin\theta)^n=2^{n+1}\cos^n(\theta/2)\cos\dfrac{n\theta}{2}$

(iv) $(1+i)^{4n}$ and $(1+i)^{4n+2}$ are real and purely imaginary respectively

(7) If $x+\dfrac{1}{x}=2\cos\theta$ prove that (i) $x^n+\dfrac{1}{x^n}=2\cos n\theta$ (ii) $x^n-\dfrac{1}{x^n}=2i\sin n\theta$

(9) If $x=\cos\alpha+i\sin\alpha$; $y=\cos\beta+i\sin\beta$ prove that $x^my^n+\dfrac{1}{x^my^n}=2\cos(m\alpha+n\beta)$

EXERCISE 3.5.

(1) Find all the values of the following:

(ii) $(8i)^{1/3}$

(2) If $x = a + b$, $y = a\omega + b\omega^2$, $z = a\omega^2 + b\omega$ show that

(i) $xyz = a^3 + b^3$

(ii) $x^3 + y^3 + z^3 = 3(a^3 + b^3)$ where ω is the complex cube root of unity.

(3) Prove that if $\omega^3 = 1$, then

(i) $(a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega) = a^3+b^3+c^3-3abc$

(iii) $\dfrac{1}{1+2\omega} - \dfrac{1}{1+\omega} + \dfrac{1}{2+\omega} = 0$

(4) Solve: (i) $x^4 + 4 = 0$

Example 3.9:

Find the modulus and argument of the following complex numbers:

(i) $-\sqrt{2} + i\sqrt{2}$ (ii) $1 + i\sqrt{3}$ (iii) $-1 - i\sqrt{3}$

Example 3.10:

If $(a_1 + ib_1)(a_2 + ib_2)\ldots(a_n + ib_n) = A + iB$, prove that (i) $(a_1^2 + b_1^2)(a_2^2 + b_2^2)\ldots(a_n^2 + b_n^2) = A^2 + B^2$

(ii) $\tan^{-1}\left(\dfrac{b_1}{a_1}\right) + \tan^{-1}\left(\dfrac{b_2}{a_2}\right) + \ldots + \tan^{-1}\left(\dfrac{b_n}{a_n}\right) = k\pi + \tan^{-1}\left(\dfrac{B}{A}\right), \quad k \in Z$

Example 3.13:

Prove that the complex numbers $3+3i$, $-3-3i$, $-3\sqrt{3}+3\sqrt{3}\,i$ are the vertices of an equilateral triangle in the complex plane.

Example 3.14:

Prove that the points representing the complex numbers $2i$, $1+i$, $4+4i$ and $3+5i$ on the Argand plane are the vertices of a rectangle.

Example 3.15:

Show that the points representing the complex numbers $7+9i$, $-3+7i$, $3+3i$ form a right angled triangle on the Argand diagram.

Example 3.16:

Find the square root of $(-7+24i)$

Example 3.17:

Solve the equation $x^4 - 4x^2 + 8x + 35 = 0$, if one of its roots is $2+\sqrt{3}\,i$

Example 3.19:

Simplify : $\dfrac{(\cos\theta + i\sin\theta)^4}{(\sin\theta + i\cos\theta)^5}$

Example 3.20:

If n is a positive integer , prove that $\left(\dfrac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n = \cos n\left(\dfrac{\pi}{2}-\theta\right) + i\sin n\left(\dfrac{\pi}{2}-\theta\right)$

Example 3.21:

If n is a positive integer, prove that $(\sqrt{3}+i)^n + (\sqrt{3}-i)^n = 2^{n+1}\cos\dfrac{n\pi}{6}$

(1) State and prove the triangle inequality of complex numbers.

(2) For any two complex numbers Z_1, Z_2, show that

(i) $|Z_1 Z_2| = |Z_1||Z_2|$ (ii) $\arg(Z_1 Z_2) = \arg(Z_1) + \arg(Z_2)$

(3) For any two complex numbers Z_1 , Z_2 , show that

(i) $\left|\frac{Z_1}{Z_2}\right| = \frac{|Z_1|}{|Z_2|}$ (ii) $\arg\left(\frac{Z_1}{Z_2}\right) = \arg(Z_1) - \arg(Z_2)$

(4) Show that for any polynomial equation $P(x)=0$ with real coefficients imaginary roots occur in conjugate pairs.

EXERCISE 4.1.

(2) Find the axis, vertex ,focus equation of directrix, latus rectum, length of the latus rectum fro the following parabolas and hence sketch their graphs.

(iii) $(x-4)^2 = 4(y+2)$

(3) If a parabolic reflector is 20cm in diameter and 5cm deep, find the distance of the focus from the centre of the reflector.

(4) The focus of a parabolic mirror is at a distance of 8cm from its centre (vertex). If the mirror is 25cm deep, find the diameter of the mirror.

EXERCISE 4.2.

(1) Find the equation of the ellipse if

(ii) the foci are (2,-1) , (0,-1) and e = 1 / 2.

(iii) the foci are $(\pm 3, 0)$ and the vertices are $(\pm 5, 0)$

(iv) the centre is (3,-4) , one of the foci is $(3+\sqrt{3}, -4)$ and $e = \frac{\sqrt{3}}{2}$

(v) the centre at the origin , the major axis is along x axis , e = 2 / 3 and passes through the point $\left(2, \frac{-5}{3}\right)$

(vi) the length of the semi major axis, and the latus rectum are 7 and 80/7 respectively, the centre is (2,5) and the major axis is parallel to y-axis.

(vii) the centre is (3,-1) , one of the foci is (6,-1) and passing through the point (8,-1).

(viii) the foci are $(\pm 3, 0)$, and the length of the latus rectum is 32 / 5 .

(ix) the vertices are $(\pm 4, 0)$ and $e = \frac{\sqrt{3}}{2}$

(3) Find the locus of a point which moves so that the sum of its distances from (3,0) and (-3,0) is 9.

(4) Find the equations and length of major and minor axes of

(ii) $5x^2 + 9y^2 + 10x - 36y - 4 = 0$ (iv) $16x^2 + 9y^2 + 32x - 36y - 92 = 0$

(5) Find the equations of directrices , latus rectum and length of latus rectums of the following ellipses:

(iii) $x^2 + 4y^2 - 8x - 16y - 68 = 0$ (iv) $3x^2 + 2y^2 - 30x - 4y + 23 = 0$

EXERCISE 4.3.

(1) Find the equatin of the hyperbola if

(i) focus : (2,3) ; corresponding directrix : $x + 2y = 5,\ e = 2$

(ii) Centre : (0,0) ; length of the semi-transverse axis is 5 ; e = 7 / 5 the conjugate axis is along x-axis.

(iii) centre : (0,0) length of the semi – transverse axis is 6 ; e = 3, transverse axis is parallel to y-axis.

(iv) centre : (1,-2) ; length of the transverse axis is 8 ; e = 5 / 4transverse axis is parallel to x-axis.

(v) centre ; (2,5) ; the distance between the directrices is 15, the distance between the foci is 20 and the transverse axis is parallel to y – axis.

(vi) foci : $(0, \pm 8)$; length of transverse axis is 12

(vii) foci : $(\pm 3, 5)$; e = 3

(viii) centre : (1,4) ; one of the foci (6,4) and the corresponding directrix is x = 9 / 4 .

(ix) foci : (6,-1) and (-4,-1) and passing through the point (4,-1)

(2) Find the equation and length of transverse and conjugate axes of the following hyperbolas:
(iii) $16x^2 - 9y^2 + 96x + 36y - 36 = 0$

(3) Find the equations of directrices, latus rectums and length of latus rectum of the following hyperbolas.
(ii) $9x^2 - 4y^2 - 36x + 32y + 8 = 0$

(4) Show that the locus of a point which moves so that the difference of its distances from the points (5,0) and (-5,0) is 8 is $9x^2 - 16y^2 = 144$.

EXERCISE 4.4.

(1) Find the equations of the tangent and normal to the parabolas.
(i) $y^2 = 12x$ at $(3, -6)$ (ii) $x^2 = 9y$ at $(-3, 1)$
(iii) $x^2 + 2x - 4y + 4 = 0$ at $(0, 1)$ (iv) to the ellipse $2x^2 + 3y^2 = 6$ at $(\sqrt{3}, 0)$
(v) to the hyperbola $9x^2 - 5y^2 = 31$ at $(2, -1)$

(2) Find the equations of the tangent and normal
(i) to the parabola $y^2 = 8x$ at $t = \frac{1}{2}$
(ii) to the ellipse $x^2 + 4y^2 = 32$ at $\theta = \frac{\pi}{4}$
(iii) to the ellipse $16x^2 + 25y^2 = 400$ at $t = \frac{1}{\sqrt{3}}$
(iv) to the hyperbola $\frac{x^2}{9} - \frac{y^2}{12} = 1$ at $\theta = \frac{\pi}{6}$

(3) Find the equations of the tangents
(i) to the parabola $y^2 = 6x$, parallel to $3x - 2y + 5 = 0$
(ii) to the parabola $y^2 = 16x$, perpendicular to the line $3x - y + 8 = 0$
(iii) to the ellipse $\frac{x^2}{20} + \frac{y^2}{5} = 1$, which are perpendicular to $x + y + 2 = 0$
(iv) to the hyperbola $4x^2 - y^2 = 64$, which are parallel to $10x - 3y + 9 = 0$

(4) Find the equations of the two tangents that can be drawn
(i) from the point (2,-3) to the parabola $y^2 = 4x$
(ii) from the point (1,3) to the ellipse $4x^2 + 9y^2 = 36$
(iii) from the point (1,2) to the hyperbola $2x^2 - 3y^2 = 6$.

EXERCISE 4.5.

(1) Find the equation of the asymptotes to the hyperbola
(ii) $8x^2 + 10xy - 3y^2 - 2x + 4y - 2 = 0$

(2) Find the equation of the hyperbola if
(i) the asymptotes are $2x + 3y - 8 = 0$ and $3x - 2y + 1 = 0$ and (5,3) is a point on the hyperbola.

(3) Find the angle between the asymptotes of the hyperbola
(iii) $4x^2 - 5y^2 - 16x + 10y + 31 = 0$

EXERCISE 4.6.

(1) Find the equation of the rectangular hyperbola whose centre is $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ and which passes through the point $\left(1, \frac{1}{4}\right)$.

(2) Find the equation of the tangent and normal (i) at (3,4) to the rectangular hyperbolas $xy = 12$ (ii) at $\left(-2, \frac{1}{4}\right)$ to the rectangular hyperbola $2xy - 2x - 8y - 1 = 0$

(4) A standard rectangular hyperbola has its vertices at(5,7) and (-3,-1). Find its equation and asymptotes.

(5) Find the equation of the rectangular hyperbola which has its centre at (2,1) one of its asymptotes $3x - y - 5 = 0$ and which passes through the point (1,-1).

(6) Find the equations of the asymptotes of the following rectangular hyperbolas.

(ii) $2xy + 3x + 4y + 1 = 0$ (iii) $6x^2 + 5xy - 6y^2 + 12x + 5y + 3 = 0$

(7) Prove that the tangent at any point to the rectangular hyperbola forms with the asymptotes a triangle of constant area.

Example 4.7 :

Find the axis, vertex, focus, directrix, equation of the latus rectum, length of the latus rectum for the following parabolas and hence draw their graphs.

(iii) $(y+2)^2 = -8(x+1)$

Example 4.9:

The headlight of a motor vehicle is a parabolic reflector of diameter 12cm and depth 4cm. Find the position of bulb on the axis of the reflector for effective functioning of the headlight.

Example 4.11:

A reflecting telescope has a parabolic mirror for which the distance from the vertex to the focus is 9mts. If the distance across (diameter) the top of the mirror is 160cm, how deep is the mirror at the middle?

Example 4.15:

Find the equation of the ellipse whose foci are (1,0) and (-1,0) and eccentricity is 1 / 2.

Example 4.16:

Find the equation of the ellipse whose one of the foci is (2,0) and the corresponding directrix is x=8 and eccentricity is 1 / 2.

Example 4.17:

Find the equation of the ellipse with focus (-1,-3), directrix $x - 2y = 0$ and eccentricity 4 / 5.

Example 4.18:

Find the equation of the ellipse with foci $(\pm 4, 0)$ and vertices $(\pm 5, 0)$

Example 4.20:

Find the equation of the ellipse whose centre is (1,2) , one of the foci is (1,3) and eccentricity is 1 / 2.

Example 4.21:

Find the equation of the ellipse whose major axis is along x-axis, centre at the origin,passes through the point (2,1) and eccentricity 1 / 2.

Example 4.22:

Find the equation of the ellipse if the major axis is parallel to y-axis, semi – major axis is 12, length of the latus rectum is 6 and the centre is (1,12).

Example 4.23:

Find the equation of the ellipse given that the centre is (4,-1) , focus is (1,-1) and passing through (8,0)

Example 4.24:

Find the equation of the ellipse whose foci are (2,1) (-2,1) and length of the latus rectum is 6.

Example 4.25:

Find the equation of the ellipse whose vertices are (-1,4) and (-7,4) and eccentricity is 1 / 3.

Example 4.26:

Find the equation of the ellipse whose foci are (1,3) and (1,9) and eccentricity is 1 / 2.

Example 4.27:

Find the equation of a point which moves so that the sum of its distances from (-4,0) and (4,0) is 10.

Example 4.28:

Find the equations and lengths of major and minor axes of

(iii) $\frac{(x-1)^2}{9}+\frac{(y+1)^2}{16}=1$

Example 4.29:

Find the equations of axes and length of axes of the ellipse $6x^2+9y^2+12x-36y-12=0$

Example 4.30:

Find the equations of directrices, latus rectum and length of latus rectum of the following ellipses.

(iii) $4x^2+3y^2+8x+12y+4=0$

Example 4.31:

Find the eccentricity, centre, foci, vertices of the following ellipse :

(iii) $\frac{(x+3)^2}{6}+\frac{(y-5)^2}{4}=1$

Example 4.34:

The orbit ofthe earth around the sun is elliptical in shape with sun at a focus. The semi major axis is of length 92.9 million miles and eccentricity is 0.017. Find how close the earth gets to sun and the greatest possible distance between the earth and the sun.

Example 4.36:

Find the equation of hyperbola whose directrix is $2x+y=1$, focus (1,2) and eccentricity $\sqrt{3}$.

Example 4.37:

Find the equation of the hyperbola whose transverse axis is along x-axis. The centre is (0,0) length of semi-transverse axis is 6 and eccentricity is 3.

Example 4.38:

Find the equation of the hyperbola whose transverse axis is parallel to x-axis, centre is (1,2) , length of the conjugate axis is 4 and eccentricity e = 2.

Example 4.39:

Find the equation of the hyperbola whose centre is (1,2). The distance between the directrices is 20 / 3, the distance between the foci is 30 and the transverse axis is parallel to y – axis.

Example 4.41:

Find the equation of the hyperbola whose foci are $(\pm 6, 0)$ and length of the transverse axis is 8.

Example 4.42:

Find the equation of the hyperbola whose foci are $(5, \pm 4)$ and eccentricity is 3 / 2.

Example 4.43:

Find the equation of the hyperbola whose centre is (2,1) , one of the foci are (8,1) and the corresponding directrix is x =4.

Example 4.44:

Find the equation of the hyperbola whose foci are $(0, \pm 5)$ and the length of the transverse axis is 6.

Example 4.45:

Find the equation of the hyperbola whose foci are $(0, \pm\sqrt{10})$ and passing through (2,3).

Example 4.48:

Find the equations and length of transverse and conjugate axes of the hyperbola $9x^2-36x-4y^2-16y+56=0$.

Example 4.51:

Find the equations of directrices, latus rectum and length of latus rectum of the hyperbola $9x^2 - 36x - 4y^2 - 16y + 56 = 0$

Example 4.52:

The foci of a hyperbola coincide with the foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$. Determine the equation of the hyperbola if its eccentricity is 2.

Example 4.53:

Find the equation of the locus of all points such that the differences of their distances from (4,0) and (-4,0) is always equal to 2.

Example 4.54:

Find the eccentricity, centre, foci and vertices of the hyperbola $\frac{x^2}{4} - \frac{y^2}{5} = 1$ and also trace the curve.

Example 4.55:

Find the eccentricity , centre , foci and vertices of the hyperbola $\frac{y^2}{6} - \frac{x^2}{18} = 1$ and also trace the curve.

Example 4.58:

Points A and B are 10km apart and it is determined from the sound of an explosion heard at those points at different times that the location of the explosion is 6km closer to A than B. Show that the location of the explosion is restricted to a particular curve and find an equation of it.

Example 4.59:

Find the equations of the tangents to the parabola $y^2 = 5x$ from the point (5,13). Also find the points of contact.

Example 4.61:

Find the equation of the tangent and normal to the parabola $x^2 + x - 2y + 2 = 0$ at (1,2).

Example 4.62:

Find the equations of the two tangents that can be drawn from the point (5,2) to the ellipse $2x^2 + 7y^2 = 14$

Example 4.64:

Find the separate equations of the asymptotes of the hyperbola $3x^2 - 5xy - 2y^2 + 17x + y + 14 = 0$

Example 4.65:

Find the equation of the hyperbola which passes through the point (2,3) and has the asymptotes $4x + 3y - 7 = 0$ and $x - 2y = 1$.

Example 4.66:

Find the angle between the asymptotes of the hyperbola $3x^2 - y^2 - 12x - 6y - 9 = 0$

Example 4.67:

Find the angle between the asymptotes to the hyperbola $3x^2 - 5xy - 2y^2 + 17x + y + 14 = 0$

Example 4.68:

Prove that the product of perpendiculars from any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ to its asymptotes is constant and the value is $\frac{a^2b^2}{a^2 + b^2}$

Example 4.69:

Find the equation of the standard rectangular hyperbola whose centre is $\left(-2, \frac{-3}{2}\right)$ and which passes through the point $\left(1, \frac{-2}{3}\right)$

Example 4.70:

The tangent at any point of the rectangular hyperbola $xy = c^2$ makes intercepts a, b and the normal at the point makes intercepts p,q on the axes. Prove that $ap + bq = 0$

Example 4.71:

Show that the tangent to a rectangular hyperbola terminated by its asymptotes is bisected at the point of contact.

EXERCISE 5.1.

(2) A Particle of unit mass moves so that displacement after t secs is given by $x = 3 \cos(2t - 4)$. Find the acceleration and kinetic energy at the end of 2 secs. $\left[K.E. = \frac{1}{2} m v^2 , \ m \ is \ mass \right]$

(4) Newton's law of cooling is given by $\theta = \theta_0^{\circ} e^{-kt}$, where the excess of temperature at zero time is $\theta_0^{\circ} C$ and at time t seconds is $\theta^{\circ} C$. Determine the rate of change of temperature after 40 s, given that $\theta_0 = 16^{\circ} C$ and $k = -0.03$. $\left(e^{1.2} = 3.3201 \right)$

(7) Two sides of a triangle are 4m and 5m in length and the angle between them is increasing at a rate of 0.06 rad / sec. Find the rate at which the area of the triangle is increasing when the angle between the sides of fixed length is $\pi / 3$.

Example 5.10 :

Find the equations of the tangents and normal to the curve $y = x^3$ at the point (1,1).

Example 5.11 :

Find the equations of the tangent and normal to the curve $y = x^2 - x - 2$ at the point (1, -2).

Example 5.12 :

Find the equation of the tangent at the point (a,b) to the curve $xy = c^2$.

Example 5.16 :

Find the equation of the tangent to the parabola, $y^2 = 20 \ x$ which forms an angle $45°$ with the x – axis.

Example 5.19 :

Show that $x^2 - y^2 = a^2$ and $xy = c^2$ cut orthogonally.

EXERCISE 5.2.

(1) Find the equation of the tangent and normal to the curves

(i) $y = x^2 - 4x - 5$ at $x = -2$ (ii) $y = x - \sin x \cos x$, at $x = \frac{\pi}{2}$

(iii) $y = 2\sin^2 3x$ at $x = \frac{\pi}{6}$ (iv) $y = \frac{1 + \sin x}{\cos x}$ at $x = \frac{\pi}{4}$

(2) Find the points on curve $x^2 - y^2 = 2$ at which the slope of the tangent is 2.

(3) Find at what points on the circle $x^2 + y^2 = 13$, the tangent is parallel to the line $2x + 3y = 7$

(4) At what points on the curve $x^2 + y^2 - 2x - 4y + 1 = 0$ the tangent is parallel to (i) x–axis (ii) y – axis.

(6) Find the equation of a normal to $y = x^3 - 3x$ that is parallel to $2x + 18y - 9 = 0$.

(8) Prove that the curve $2x^2 + 4y^2 = 1$ and $6x^2 - 12y^2 = 1$ cut each other at right angles.

(9) At what angle θ do the curves $y = a^x$ and $y = b^x$ intersect $(a \neq b)$?

Example 5.22 :

Verify Rolle's theorem for the following :

(v) $f(x) = e^x \sin x$, $0 \leq x \leq \pi$

Example 5.23 :

Apply Rolle's theorem to find points on curve $y = -1 + \cos x$, where the tangent is parallel to x – axis in $[0, 2\pi]$.

EXERCISE 5.3.

(2) Using Rolle's theorem find the points on the curve $y = x^2 + 1, \quad -2 \le x \le 2$ where the tangent is parallel to x – axis.

Example 5.24 :

Verify Lagrange's law of the mean for $f(x) = x^3$ on $[-2, 2]$

Example 5.25 :

A cylindrical hole 4mm in diameter and 12 mm deep in a metal block is rebored to increase the diameter to 4.12 mm. Estimate the amount of metal removed.

Example 5.26 :

Suppose that $f(0) = -3$ and $f'(x) \le 5$ for all values of x, how large can $f(2)$ possibly be?

Example 5.27 :

It took 14 sec for a thermometer to rise from -19° C to 100°C when it was taken from a freezer and placed in boiling water. Show that somewhere along the way the mercury was rising at exactly 8.5 ° C / sec.

EXERCISE 5.4

(1) Verify Lagrange's law of mean for the following functions :

(i) $f(x) = 1 - x^2$, $[0,3]$

(ii) $f(x) = \frac{1}{x}$, $[1,2]$

(iii) $f(x) = 2x^3 + x^2 - x - 1$, $[0,2]$

(iv) $f(x) = x^{2/3}$, $[-2,2]$

(v) $f(x) = x^3 - 5x^2 - 3x$, $[1,3]$

(2) If $f(1) = 10$ and $f'(x) \ge 2$ for $1 \le x \le 4$ how small can $f(4)$ possibly be ?

(3) At 2.00 p.m. a car's speedometer reads 30 miles / hr., at 2.10 pm it reads 50 miles / hr. Show that sometime between 2.00 and 2.10 the acceleration is exactly 120 miles / hr^2.

Example 5.28 :

Obtain the Maclaurin's Series for

(2) $\log_e (1 + x)$

(3) arc $\tan x$ *or* $\tan^{-1} x$

EXERCISE 5.5.

(1) Obtain the Maclaurin's Series expansion for :

(ii) $\cos^2 x$

(iii) $\frac{1}{1+x}$

(iv) $\tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

Example 5.30 :

Find $\lim\limits_{x \to +\infty} \dfrac{\sin \frac{1}{x}}{\tan^{-1} \frac{1}{x}}$ if exists.

Example 5.31 : $\lim\limits_{x \to \pi/2} \dfrac{\log(\sin x)}{(\pi - 2x)^2}$

Example 5.33 : Evaluate : $\lim_{x \to 0} \left(\cos ec\, x - \frac{1}{x} \right)$

Example 5.36 :

The current at time t in a coil with resistance R, inductance L and subjected to a constant electromotive force E is given by $i = \frac{E}{R}\left(1 - e^{\frac{-RT}{L}}\right)$ Obtain a suitable formula to be used when R is very small.

EXERCISE 5.6.

Evaluate the limit for the following if exists .

(2) $\lim_{x \to 0} \frac{\tan x - x}{x - \sin x}$

(6) $\lim_{x \to \infty} \frac{\frac{1}{x^2} - 2\tan^{-1}\left(\frac{1}{x}\right)}{\frac{1}{x}}$

(8) $\lim_{x \to 0} \frac{\cot x}{\cot 2x}$

(9) $\lim_{x \to 0+} x^2 \log_e x.$

(10) $\lim_{x \to 1} x^{\frac{1}{x-1}}$

(12) $\lim_{x \to 0+} x^x$

(13) $\lim_{x \to 0} (\cos x)^{1/x}$

Example 5.37:

Prove that the function $f(x) = \sin x + \cos 2x$ is not monotonic on the interval $\left[0, \frac{\pi}{4}\right]$

Example 5.38:

Find the intervals in which $f(x) = 2x^3 + x^2 - 20x$ is increasing and decreasing.

Example 5.39 :

Prove that the function $f(x) = x^2 - x + 1$ is neither increasing nor decreasing in $[0,1]$

Example 5.40 :

Discuss monotonicity of the function $f(x) = \sin x,\ x \in [0, 2\pi]$

Example 5.41:

Determine for which values of x, the function $y = \frac{x-2}{x+1}$, $x \neq -1$ is strictly increasing or strictly decreasing.

Example 5.42 :

Determine for which values of x, the function $f(x) = 2x^3 - 15x^2 + 36x + 1$ is increasing and for which it is decreasing. Also determine the points where the tangents to the graph of the function are parallel to the x axis.

Example 5.43 :

Show that $f(x) = \tan^{-1}(\sin x + \cos x),\ x > 0$ is a strictly increasing function in the interval $\left(0, \frac{\pi}{4}\right)$

EXERCISE 5.7

(3) Which of the following functions increasing or decreasing on the interval given ?

(iv) $x(x-1)(x+1)$ *on* $[-2,-1]$

(v) $x \sin x$ *on* $\left[0, \frac{\pi}{4}\right]$

(4) Prove that the following functions are not monotonic in the intervals given.

(i) $2x^2 + x - 5$ on $[-1,0]$

(ii) $x(x-1(x+1))$ *on* $[0,2]$

(iii) $x \sin x$ *on* $[0, \pi]$

(iv) $\tan x + \cot x$ *on* $\left[0, \frac{\pi}{2}\right]$

(5) Find the intervals on which f is increasing or decreasing.

(i) $f(x) = 20 - x - x^2$

(ii) $f(x) = x^3 - 3x + 1$

(iii) $f(x) = x^3 + x + 1$

(iv) $f(x) = x - 2\sin x, \quad [0, 2\pi]$

(v) $f(x) = x + \cos x$ in $[0, \pi]$

(vi) $f(x) = \sin^4 x + \cos^4 x$ in $\left[0, \frac{\pi}{2}\right]$

Example 5.45 :

Prove that the inequality $(1+x)^n > 1 + nx$ is true whenever $x > 0$ and $n > 1$.

Example 5.46:

Prove that $\sin x < x < \tan x, \quad x \in \left(0, \frac{\pi}{2}\right)$

EXERCISE 5.8.

(1) Prove the following inequalities :

(i) $\cos x > 1 - \frac{x^2}{2}, \quad x > 0$

(ii) $\sin x > x - \frac{x^3}{6}, \quad x > 0$

(iii) $\tan^{-1} x < x$ for all $x > 0$

(iv) $\log(1+x) < x$ for all $x > 0$.

Example 5.48 :

Find the absolute maximum and minimum values of the function. $f(x) = x^3 - 3x^2 + 1, \quad -\frac{1}{2} \le x \le 4$

Example 5.49 :

Discuss the curve $y = x^4 - 4x^3$ with respect to local extrema.

Example 5.50 :

Locate the extreme point on the curve $y = 3x^2 - 6x$ and determine its nature by examine the sign of the gradient on either side.

EXERCISE 5.9.

(1) Find the critical numbers and stationary points of each of the following functions .

(iii) $f(x) = x^{4/5}\,(x-4)^2$

(iv) $f(x) = \dfrac{x+1}{x^2+x+1}$

(vi) $f(\theta) = \sin^2 2\theta$ in $[0, \pi]$

(vii) $f(\theta) = \theta + \sin\theta$ in $[0, 2\pi]$

(2) Find the absolute maximum and absolute minimum values of f on the given interval:

(i) $f(x) = x^2 - 2x + 2$, $[0,3]$

(ii) $f(x) = 1 - 2x - x^2$, $[-4,1]$

(iii) $f(x) = x^3 - 12x + 1$, $[-3,5]$

(iv) $f(x) = \sqrt{9-x^2}$, $[-1,2]$

(v) $f(x) = \dfrac{x}{x+1}$, $[1,2]$

(vi) $f(x) = \sin x + \cos x$, $\left[0, \dfrac{\pi}{3}\right]$

(vii) $f(x) = x - 2\cos x$, $[-\pi, \pi]$

(3) Find the local maximum and minimum values of the following functions:

(i) $x^3 - x$ (ii) $2x^3 + 5x^2 - 4x$

EXERCISE 5.10.

(1) Find the numbers whose sum is 100 and whose product is a maximum.

(2) Find two positive numbers whose product is 100 and whose sum is minimum.

(6) Resistance to motion F, of a moving vehicle is given by $F = \dfrac{5}{x} + 100x$. Determine the minimum value of resistance.

Example 5.62 :

Determine where the curve $y = x^3 - 3x + 1$ is cancave upward and where it is concave downward. Also find the inflection points.

Example 5.65 :

Determine the points of inflection if any, of the function $y = x^3 - 3x + 2$

Example 5.66 :

Test for points of inflection of the curve $y = \sin x$, $x \in (0, 2\pi)$

EXERCISE 5.11.

Find the intervals of concavity and the points of inflection of the following functions:

(1) $f(x) = (x-1)^{1/3}$ (3) $f(x) = 2x^3 + 5x^2 - 4x$

Example 6.2 :

Compute the values of Δy and dy if $y = f(x) = x^3 + x^2 - 2x + 1$ where x changes (i) from 2 to 2.05 and (ii) from 2 to 2.01.

Example 6.3 :

Use differentials to find an approximate value for $\sqrt[3]{65}$.

Example 6.5 :

The time of swing T of a pendulum is given by $T = k\sqrt{\ell}$ where k is a constant. Determine the percentage error in the time of swing if the length of the pendulum / changes from 32.1 cm to 32.0 cm.

Example 6.6 :

A circular template has a radius of 10 cm (± 0.02). Determine the possible error in calculating the area of the templates. Find also the percentage error.

Example 6.7.

Show that the percentage error in the n[th] root of a number is approximately $\frac{1}{n}$ times the percentage in the number.

EXERCISE 6.1.

(3) Use differentials to find an approximate value for the given number

(i) $\sqrt{36.1}$ (ii) $\frac{1}{10.1}$ (iv) $(1.97)^6$

(4) The edge of a cube was found to be 30 cm with a possible error in measurement of 0.1 cm. Use differentials to estimate the maximum possible error in computing (i) the volume of the cube and (ii) the surface area of cube.

(5) The radius of a circular disc is given as 24cm with a maximum error in measurement of 0.02cm.

(i) Use differentials to estimate the maximum error in the calculated area of the disc.

(ii) Compute the relative error?

Example 6.15:

If $u = \log(\tan x + \tan y + \tan z)$, prove that $\sum \sin 2x \frac{\partial u}{\partial x} = 2$

Example 6.16:

If $U = (x-y)(y-z)(z-x)$ then show that $U_x + U_y + U_z = 0$

Example 6.17:

Suppose that $z = ye^{x^2}$ where $x = 2t$ and $y = 1-t$ then find $\frac{dz}{dt}$

Example 6.19:

If $w = x + 2y + z^2$ and $x = \cos t$; $y = \sin t$; $z = t$. Find $\frac{dw}{dt}$.

Example 6.21:

If u is a homogenous function of x and y of degree n , prove that $x\frac{\partial^2 u}{\partial x\, \partial y} + y\frac{\partial^2 u}{\partial y^2} = (n-1)\frac{\partial u}{\partial y}$

EXERCISE 6.3.

(1) Verify $\frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial^2 u}{\partial y\, \partial x}$ for the following function:

(i) $u = x^2 + 3xy + y^2$

(3) Using chain rule find $\frac{dw}{dt}$ for each of the following:

(i) $w = e^{xy}$ where $x = t^2$, $y = t^3$

(ii) $w = \log(x^2 + y^2)$ where $x = e^t$, $y = e^{-t}$

(iii) $w = \frac{x}{(x^2 + y^2)}$ where $x = \cos t$, $y = \sin t$

(iv) $w = xy + z$ where $x = \cos t$, $y = \sin t, z = t$

(4) (i) Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial \theta}$ if $w = \log(x^2 + y^2)$ where $x = r\cos\theta$, $y = r\sin\theta$

(ii) Find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ if $w = x^2 + y^2$ where $x = u^2 - v^2$, $y = 2uv$

(iii) Find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ if $w = \sin^{-1} xy$ where $x = u + v$, $y = u - v$.

(5) Using Euler's theorem prove the following:

(ii) $u = xy^2 \sin\left(\frac{x}{y}\right)$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3u.$

(iii) If u is a homogeneous function of x and y of degree n, prove that $x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\,\partial y} = (n-1)\frac{\partial u}{\partial x}$

(iv) If $V = ze^{ax+by}$ and z is a homogenous function of degree n in x and y prove that $x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} = (ax+by+n)\,V.$

EXERCISE 7.1.

Evaluate the following problems using second fundamental theorem:

(3) $\int_0^1 \sqrt{9-4x^2}\;dx$ (4) $\int_0^{\pi/4} 2\sin^2 x \sin 2x\;dx$ (6) $\int_0^{\pi/2} \frac{\sin x\,dx}{9+\cos^2 x}$ (7) $\int_1^2 \frac{dx}{x^2+5x+6}$ (8) $\int_0^1 \frac{(\sin^{-1} x)^3}{\sqrt{1-x^2}}\;dx$ (9) $\int_0^{\pi/2} \sin 2x\;\cos x\;dx$

(10) $\int_0^1 x^2 e^x\,dx$

Example 7.7 :

Evaluate : $\int_{-\pi/2}^{\pi/2} x\sin x\,dx$

Example 7.8 :

Evaluate $\int_{-\pi/2}^{\pi/2} \sin^2 x\;dx$

Example 7.9:

Evaluate $\int_0^{\pi/2} \frac{f(\sin x)}{f(\sin x)+f(\cos x)}\;dx$

Example7.10:

Evaluate $\int_0^1 x(1-x)^n\;dx$

Example 7.11:

Evaluate $\int_0^{\pi/2} \log(\tan x)\;dx$

Example 7.12

Evaluate $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\cot x}}$

EXERCISE 7.2.

Evaluate the following Problems using properties of integration.

(3) $\int_0^{\pi/2} \sin^3 x\;\cos x\;dx$ (4) $\int_{-\pi/2}^{\pi/2} \cos^3 x\;dx$ (5) $\int_{-\pi/2}^{\pi/2} \sin^2 x\;\cos x\;dx$

(7) $\int_0^1 \log\left(\frac{1}{x}-1\right)\;dx$ (8) $\int_0^3 \frac{\sqrt{x}\,dx}{\sqrt{x}+\sqrt{3-x}}$ (9) $\int_0^1 x(1-x)^{10}\;dx$ (10) $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$

Example 7.13:

Evaluate: $\int \sin^5 x\;dx$

Example 7.14:

Evaluate $\int \sin^6 x \ dx$

Example 7.15:

Evaluate

(iii) $\int_0^{2\pi} \sin^9 \frac{x}{4} \ dx$ (iv) $\int_0^{\pi/6} \cos^7 3x \ dx$

Example 7.16:

$\int_0^{\pi/2} \sin^4 x \ \cos^2 x \ dx$

Example 7.17:

Evaluate

(i) $\int x^3 e^{2x} \ dx$ (ii) $\int_0^1 x e^{-4x} \ dx$

EXERCISE 7.3.

(1) Evaluate : (i) $\int \sin^4 x \ dx$ (ii) $\int \cos^5 x \ dx$

(3) Evaluate : (i) $\int_0^{\pi/4} \cos^8 2x \ dx$ (ii) $\int_0^{\pi/6} \sin^7 3x \ dx$

(4) Evaluate : (i) $\int_0^1 x e^{-2x} \ dx$

Example 7.18:

Find the area of the region bounded by the line $3x - 2y + 6 = 0$, $x = 1$, $x = 3$ and x-axis.

Example 7.19:

Find the area of the region bounded by the line $3x - 5y - 15 = 0$, $x = 1$, $x = 4$ and x-axis.

Example 7.20:

Find the area of the region bounded by the line $y = x^2 - 5x + 4$, $x = 2$, $x = 3$ and the x-axis.

Example 7.21:

Find the area of the region bounded by the line $y = 2x + 1$, $y = 3$, $y = 5$ and y-axis.

Example 7.22:

Find the area of the region bounded by the line $y = 2x + 4$, $y = 1$, $y = 3$ and y-axis.

Example 7.23:

(i) Evaluate the integral $\int_1^5 (x - 3) \ dx$

(ii) Find the area of the region bounded by the line $y + 3 = x$, x = 1 and x = 5

Example 7.24:

Find the area bounded by the curve $y = \sin 2x$ between the ordinates x = 0 , x = $\pi$and x – axis.

Example 7.35:

Find the volume of the solid that results when the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ \ (a > b > 0)$ is revolved about the minor axis.

Example 7.36:

Find the volume of the solid generated when the region enclosed by $y = \sqrt{x}$, $y = 2$ and $x = 0$ is revolved about the y – axis.

EXERCISE 7.4.

(1) Find the area of the region bounded by the line $x - y = 1$ and

(i) x - axis , x = 2 and x = 4 (ii) x - axis , x = -2 and x = 0

(2) Find the area of the region bounded by the line $x - 2y - 12 = 0$ and

(i) y - axis , y = 2 and y = 5 (ii) y - axis , y = -1 and y = -3

(3) Find the area of the region bounded by the line $y = x - 5$ and the x-axis between the ordinates x = 3 and x = 7

(5) Find the area of the region bounded by $x^2 = 36y$, y - axis , y = 2 and y = 4.

(6) Find the area included between the parabola $y^2 = 4ax$ and its latus rectum.

(10) Find the area of the circle whose radius is a.

Find the volume of the solid that result when the region enclosed by the given curves:

(11) $y = 1 + x^2$, $x = 1$, $x = 2$, $y = 0$ is revolved about the x – axis.

(12) $2ay^2 = x(x-a)^2$ is revolved about x - axis, a>0.

(13) $y = x^3$, $x = 0$, $y = 1$ is revolved about the y - axis.

(14) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is revolved about major axis $a > b > 0$.

(16) The area of the region bounded by the curve $xy = 1$, x - axis, $x = 1$ and $x = \infty$. Find the volume of the solid generated by revolving the area mentioned about x - axis.

Example 8.2:

Form the differential equation from the following equations.

(iii) $Ax^2 + By^2 = 1$

EXERCISE 8.1.

(2) Form the differential equations by eliminating arbitrary constants given in brackets against each

(ix) $y = Ae^{2x}\cos(3x + B)$ $\{A, B\}$

(3) Find the differential equation of the family of straight lines $y = mx + \frac{a}{m}$ when (i) m is the parameter ; (ii) a is the parameter ;

(iii) a , m both are parameters.

(4) Find the differential equation that will represent the family of all circles having centres on the x-axis and the radius is unity.

Example 8.3:

Solve : $\frac{dy}{dx} = 1 + x + y + xy$

Example 8.4:

Solve: $3e^x \tan y\, dx + (1 + e^x)\sec^2 y\, dy = 0$

Example 8.5:

Solve: $\frac{dy}{dx} + \left(\frac{1-y^2}{1-x^2}\right)^{\frac{1}{2}} = 0$

Example 8.6:

Solve: $e^x\sqrt{1-y^2}\, dx + \frac{y}{x} dy = 0$

Example 8.8:

Solve: $x\, dy = \left(y + 4x^5 e^{x^4}\right) dx$

Example 8.9:

Solve: $(x^2 - y)dx + (y^2 - x)dy = 0$, if it passes through the origin.

Example 8.11:

The normal lines to a given curve at each point (x, y) on the curve pass through the point (2,0). The curve passes through the point (2, 3). Formulate the differential equation representing the problem and hence find the equation of the curve.

EXERCISE 8.2.

Solve the following:

(1) $\sec 2x\, dy - \sin 5x\ \sec^2 y\, dx = 0$ (2) $\cos^2 x\, dy + ye^{\tan x} dx = 0$

(3) $\left(x^2 - yx^2\right)dy + \left(y^2 + xy^2\right)dx = 0$ (4) $yx^2\, dx + e^{-x}\, dy = 0$

(5) $\left(x^2 + 5x + 7\right)dy + \sqrt{9 + 8y - y^2}\, dx = 0$ (6) $\frac{dy}{dx} = \sin\,(x + y)$

(8) $y\, dx + x\, dy = e^{-xy}\, dx$ if it cuts the y-axis.

Example 8.12:

Solve : $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$

Example 8.16:

Solve: $xdy - ydx = \sqrt{x^2 + y^2}\, dx$

EXERCISE 8.3.

Solve the following:

(1) $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$ (2) $\frac{dy}{dx} = \frac{y(x - 2y)}{x(x - 3y)}$ (3) $\left(x^2 + y^2\right)dy = xy\ dx$

(4) $x^2\frac{dy}{dx} = y^2 + 2xy$ given that y = 1, when x = 1.

(6) Find the equation of the curve passing through (1,0) and which has slope $1 + \frac{y}{x}$ at (x, y).

Example 8.17:

Solve: $\frac{dy}{dx} + y\cot x = 2\cos x$

Example 8.20:

Solve: $(x+1)\frac{dy}{dx} - y = e^x\,(x+1)^2$

Example 8.21:

Solve: $\frac{dy}{dx} + 2y\ \tan x = \sin x$

EXERCISE 8.4.

Solve the following:

(1) $\frac{dy}{dx} + y = x$ (2) $\frac{dy}{dx} + \frac{4x}{x^2 + 1}y = \frac{1}{\left(x^2 + 1\right)^2}$ (4) $\left(1 + x^2\right)\frac{dy}{dx} + 2xy = \cos x$

(6) $\frac{dy}{dx} + xy = x$ (8) $(y - x)\frac{dy}{dx} = a^2$

Example 8.25:

Solve: $\left(D^2 - 13D + 12\right)y = e^{-2x}$

Example 8.26:

Solve: $\left(D^2 + 6D + 8\right)y = e^{-2x}$

Example 8.27:

Solve: $\left(D^2 - 6D + 9\right)y = e^{3x}$

Example 8.28:

Solve: $\left(2D^2+5D+2\right)y=e^{-\frac{1}{2}x}$

Example 8.29:

Solve: $\left(D^2-4\right)y=\sin 2x$

Example 8.30:

Solve: $\left(D^2+4D+13\right)y=\cos\ 3x$

Example 8.31:

Solve: $\left(D^2+9\right)y=\sin 3x$

Example 8.32:

Solve: $\left(D^2-3D+2\right)y=x$

Example 8.33:

Solve: $\left(D^2-4D+1\right)y=x^2$

EXERCISE 8.5.

Solve the following differential equations:

(1) $\left(D^2+7D+12\right)y=e^{2x}$ (2) $\left(D^2-4D+13\right)y=e^{-3x}$

(3) $\left(D^2+14D+49\right)y=e^{-7x}+4$ (4) $\left(D^2-13D+12\right)y=e^{-2x}+5e^{x}$

(5) $\left(D^2+1\right)y=0$ when x = 0 , y = 2 and when $x=\dfrac{\pi}{2}$, $y=-2$

(7) $\left(D^2+3D-4\right)y=x^2$ (8) $\left(D^2-2D-3\right)y=\sin x\ \cos x$

(9) $D^2y=-9\sin 3x$ (12) $\left(D^2+5\right)y=\cos^2 x$ (13) $\left(D^2+2D+3\right)y=\sin 2x$

(14) $\left(3D^2+4D+1\right)y=3e^{-x/3}$

Example 8.36 :

The temperature T of a cooling object drops at a rate proportional to the difference T - S, where S is constant temperature of surrounding medium. If initially $T=150°C$, find the temperature of the cooling object at any time t.

Example 9.4 :

Construct the truth table for the following statements :

(iii) $(\ p\vee q\ )\wedge(\sim q\ )$ (iv) $\sim\left((\sim P)\wedge(\sim q)\right)$

Example 9.5:

Construct the truth table for $(\ p\wedge q\ )\vee(\sim r\ )$

Example 9.6 :

Construct the truth table for $(\ p\vee q\ )\wedge\ r$

EXERCISE 9.2.

Construct the truth tables for the following statements:

(7) $(\ p\wedge q\ )\vee\ [\sim(p\wedge q)]$

(9) $(\ p\vee q\ )\vee\ r$

(10) $(\ p\wedge q\ )\vee\ r$

Example 9.7 :

Show that $\sim(\ p\vee q)\equiv(\sim P)\wedge(\sim q)$

Example 9.10 :

(i) Show that $\left((\sim P)\vee(\sim q)\right)\vee p$ is a tautology.

(ii) Show that $\left((\sim q)\wedge p\ \right)\wedge q$ is a contradiction.

Example 9.11 :

Use the truth table to determine whether the statement $((\sim P) \vee q) \vee (p \wedge (\sim q))$ is a tautology.

EXERCISE 9.3

(1) Use the truth table to establish which of the following statements are tautologies and which are contradictions .

(i) $((\sim P) \wedge q) \wedge p$

(ii) $(p \vee q) \vee (\sim (p \vee q))$

(iii) $(p \wedge (\sim q)) \vee ((\sim p) \vee q)$

(iv) $q \vee (p \vee (\sim q))$

(v) $(p \wedge (\sim p)) \wedge ((\sim q) \wedge p)$

(2) Show that $p \to q \equiv (\sim p) \vee q$

(3) Show that $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

(4) Show that $p \leftrightarrow q \equiv ((\sim p) \vee q) \wedge ((\sim q) \vee p)$

(5) Show that $\sim (p \wedge q) \equiv ((\sim p) \vee (\sim q))$

(6) Show that $p \to q$ and $q \to p$ are not equivalent.

(7) Show that $(p \wedge q) \to (p \vee q)$ is a tautology.

Example 9.12 :

Prove that $(Z, +)$ is an infinite abelian group.

Example 9.13 :

Show that $(R - \{0\}, .)$ is an infinite abelian group. Here denotes usual multiplication.

Example 9.14 :

Show that the cube roots of unity forms a finite abelian group under multiplication.

Example 9.15 :

Prove that the set of all 4^{th} roots of unity forms an abelian group under multiplication.

Example 9.16 :

Prove that $(C, +)$ is an infinite abelian group.

Example 9.17 :

Show that the set of all non-zero complex numbers is an abelian group under the usual multiplication of complex numbers.

Example 9.19:

Show that the set of all 2 X 2 non-singular matrices forms a non-abelian infinite group under matrix multiplication, (where the entries belong to R).

Example 9.20 :

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ form an abelian group, under multiplication of matrices.

(1) State and prove cancellation laws on groups.

(2) State and prove reversal law on inverse of a group.

Example 10.1 :

Find the probability mass function, and the cumulative distribution function for getting ' 3 ' s when two dice are thrown.

Example 10.4 :

A continuous random variable X follows the probability law,

$f(x) = \begin{cases} kx(1-x)^{10} & , 0 < x < 1 \\ 0 & , elsewhere \end{cases}$ Find k.

Example 10.5 :

A continuous random variable X has p.d.f. $f(x)=3x^2$, $0\le x\le 1$,Find a and b such that.

(i) $P(X\le a)=P(X>a)$ and (ii) $P(X>b)=0.05$

Example 10.6 :

If the probability density function of a random variable is given by $f(x)=\begin{cases} k(1-x^2) & , 0<x<1 \\ 0 & , elsewhere \end{cases}$

Find (i) k (ii) the distribution function of the random variable .

Example 10.8 :

If $f(x)=\begin{cases} \frac{A}{x} & , 1<x<e^3 \\ 0 & , elsewhere \end{cases}$ is a probability density function of a continuous random variable X , find $p(x>e)$

EXERCISE 10.1

(1) Find the probability distribution of the number of sixes in throwing three dice once.

(2) Two cards are drawn successively without replacement from a well shuffled pack of 52 cards. Find the probability distribution of the number of queens.

(3) Two bad oranges are accidentally mixed with ten good ones. Three oranges are drawn at random without replacement from this lot. Obtain the probability distribution for the number of bad oranges.

(4) A discrete random variable X has the following probability distributions.

X	0	1	2	3	4	5	6	7	8
P(x)	a	3a	5a	7a	9a	11a	13a	15a	17a

(i) Find the value of a (ii) Find $P(x<3)$ (iii) Find $P(3<x<7)$

(6) For the p.d.f $f(x)=\begin{cases} cx(1-x)^3 & , 0<x<1 \\ 0 & , elsewhere \end{cases}$

find (i) the constant C (ii) $P\left(x<\frac{1}{2}\right)$

(8) For the distribution function given by $F(x)=\begin{cases} 0 & x<0 \\ x^2 & 0\le x\le 1 \\ 1 & x>1 \end{cases}$

Find the density function. Also evaluate (i) $P(0.5<X<0.75)$ (ii) $P(X\le 0.5)$ (iii) $P(X>0.75)$

(10) A random variable X has a probability density function

$f(x)=\begin{cases} k & , 0<x<2\pi \\ 0 & , elsewhere \end{cases}$

Find (i) k (ii) $P\left(0<X<\frac{\pi}{2}\right)$ (iii) $P\left(\frac{\pi}{2}<X<\frac{3\pi}{2}\right)$

Example 10.11 :

Two unbiased dice are thrown together at random. Find the expected value of the total number of points shown up.

Example 10.12 :

The probability of success of an event is p and that of failure is q. Find the expected number of trials to get a first success.

Example10.13 :

An urn contains 4 white and 3 red balls. Find the probability distribution of the number of red balls in three draws when a ball is drawn at random with replacement. Also find its mean and variance.

Example 10.14 :

A game is played with a single fair die. A player wins Rs. 20 if a 2 turns up, Rs. 40 if a turns up, loses Rs. 30 if a 6 turns up. While he neither wins nor loses if any other face turns up. Find the expected sum of money he can win.

Example 10.15 :

In a continuous distribution the p.d.f of X is

$f(x) = \begin{cases} \frac{3}{4}x(2-x) & ,0<x<2 \\ 0 & , \ otherwise \end{cases}$ Find the mean and the variance of the distribution.

Example 10.16:

Find the mean and variance of the distribution $f(x) = \begin{cases} 3e^{-3x} & , \ 0<x<\infty \\ 0 & , \ elsewhere \end{cases}$

EXERCISE 10.2

(1) A die is tossed twice. A success is getting an odd number on a toss. Find the mean and the variance of the probability distribution of the number of successes.

(3) In an entrance examination a student has to answer all the 120 questions. Each question has four options and only one option is correct. A student gets 1 mark for a correct answer and loses half mark for a wrong answer. What is the expectation of the mark scored by a student if he chooses the answer to each question at random

(4) Two cards are drawn with replacement from a well shuffled deck of 52 cards. Find the mean and variance for the number of aces.

(5) In a gambling game a man wins Rs. 10 if he gets all heads or all tails and loses Rs. 5 if he gets 1 or 2 heads when 3 coins are tossed once. Find his expectation of gain.

(6) The probability distribution of a random variable X is given below:

X	0	1	2	3
P (X =x)	0.1	0.3	0.5	0.1

If $Y = X^2 + 2X$ find the mean and variance of Y.

(7) Find the Mean and Variance for the following probability density functions:

(i) $f(x) = \begin{cases} \frac{1}{24} & , -12 \le x \le 12 \\ 0 & , \ otherwise \end{cases}$

(ii) $f(x) = \begin{cases} \alpha e^{-\alpha x} & , if \ x > 0 \\ 0 & , \ otherwise \end{cases}$

(iii) $f(x) = \begin{cases} x e^{-x} & , if \ x > 0 \\ 0 & , \ otherwise \end{cases}$

Example 10.17 :

Let X be a binomially distributed variable with mean 2 and standard deviation $\frac{2}{\sqrt{3}}$. Find the corresponding probability function.

Example 10.18 :

A pair of dice is thrown 10 times. If getting a doublet is considered a success find the probability of (i) 4 success (ii) No success.

EXERCISE 10.3

(4) Four coins are tossed simultaneously. What is the probability of getting (a) exactly 2 heads (b) at least two heads (c) at most two heads.

(5) The overall percentage of passes in a certain examination is 80. If 6 candidates appear in the examination what is the probability that at least 5 pass the examination.

Example 10.23 :

If a publisher of non-technical books takes a great pain to ensure that his books are free of typological errors, so that the probability of any given page containing atleast one such error is 0.005 and errors are independent from page to page (i) what is the probability that one of Rs. 400 page novels will contain exactly one page with error. (ii) atmost three pages with errors. $\left[e^{-2} = 0.1363 \ ; \ e^{-0.2} = 0.819\right]$.

Example 10.24 :

Suppose that the probability of suffering a side effect from a certain vaccine is 0.005. If 1000 persons are inoculated. Find approximately the probability that (i) atmost 1 person suffer. (ii) 4,5 or 6 persons suffer. $\left[e^{-5} = 0.0067\right]$.

Example 10.25 :

In a Poisson distribution if $P(X=2) = P(X=3)$ find $P(X=5)$ $\left[given \quad e^{-3} = 0.050\right]$.

EXERCISE 10.4

(3) 20% of the bolts produced in a factory are found to be defective. Find the probability that in a sample of 10 bolts chosen at random exactly 2 will be defective using (i) Binomial distribution (ii) Poisson distribution. $\left[e^{-2} = 0.1353\right]$.

(4) Alpha particles are emitted by a radio active source at an average rate of 5 in a 20 minutes interval. Using Poisson distribution find the probability that there will be (i) 2 emission (ii) at least 2 emission in a particular 20 minutes interval. $\left[e^{-5} = 0.0067\right]$.

EXERCISE 10.5

(3) Suppose that the amount of cosmic radiation to which a person is exposed when flying by jet across the United States is a random variable having a normal distribution with a mean of 4.35 m rem and a standard deviation of 0.59 m rem. What is the probability that a person will be exposed to more than 5.20 m rem of cosmic radiation of such a flight.

(4) The life of army shoes is normally distributed with mean 8 months and standard deviation 2 months. If 5000 pairs are issued, how many pairs would be expected to need replacement within 12 months.

(6) If the height of 300 students are normally distributed with mean 64.5 inches and standard deviation 3.3 inches. Fin dthe height below which 99% of the student lie.

(7) Marks in an aptitude test given to 800 students of a school was found to be normally distributed. 10% of the students scored below 40 marks and 10% of the students scored above 90 marks. Find the number of students scored between 40 and 90.